VŠB - Technical University of Ostrava
Faculty of Mining and Geology

DESCRIPTIVE GEOMETRY

Eva Vavříková

Ostrava
2005

PREFACE

This textbook was written for the basic course of the Constructive Geometry and represents the learning tool for study of the very frequently used projection methods – Monge Projection and the Orthogonal Axonometry and their elementary constructions. Projection methods enable to create a plane representation of space objects and transform a spatial problem to a plane. The first chapter is focussed on the conic sections and their basic properties. Ellipse is necessary for other constructions, so this part is much more detailed.

I wisch to thank Doc. RNDr. Pavel Burda CSc., Mgr. Dagmar Dlouhá Ph.D., RNDr. Milan Tůma and Mgr. Jiří Vrbický, Ph.D. for the critical reading and helpfull suggestions.

RNDr. Eva Vavříková

Ostrava, August 2005

NOTATION

To facilitate the orientation in the text of the textbook, we give an overview of the most frequently used notation.

Notation:

A, B, C,...	points
a, b, c,...	straight lines
$\alpha, \beta, \rho, \ldots$	planes
$\angle\varphi, \angle ABC, \angle kh$	angles
$\pi = (x, y)$	plane defined by a pair of lines
$A \in a, B \in \rho,\ a \subset \rho, M \notin p$	symbols for incidence
$g \parallel x, a \parallel \alpha$	parallelism of lines, planes
$a \perp b,\ a \perp \pi$	perpendicularity of lines, planes
$P = b \cap c,$	intersection point of two lines
$x = \pi \cap \nu$	intersection line of two planes
$k = (S; r)$	circle with the centre S and the radius r
$\lvert AB\rvert, \lvert Ap\rvert, \lvert A\rho\rvert$	distance of two points, a point and a line etc.

CONTENTS

1. CONIC SECTIONS

Conic section is an intersection of a cone with a plane. A plane containing a vertex of a cone gives us a singular conic, a plane not containing a vertex of a cone gives us a regular conic.

Conic sections are: a circle (C), an ellipse(E), a hyperbola(H) and a parabola(P). C, E and H are called **central conic sections**.

The tangent line of a conic section is a straight line that has only one point (point of contact) common with a conic section. The other points of the tangent line are external points with respect to a conic section.

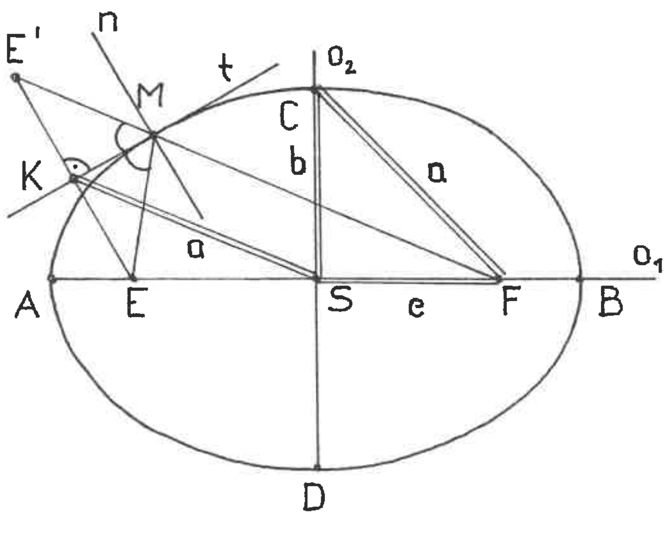
The normal is the straight line passing through the point of contact M of the tangent line and it is perpendicular to the tangent line at this point.

REGULAR CONIC SECTIONS

1.1 ELLIPSE

Ellipse is the set of all points M in the plane whose distances from two fixed points E, F, $E \neq F$ (foci of an ellipse) have a constant sum $2a, 2a > |EF|$. See Fig. 1.1.

Fig. 1.1



E, F ... focus
A, B ... major vertex
C, D ... minor vertex
S ... centre
$AB = o_1$ ... major axis, $|AS| = |BS| = |EC| = |FC| = ... = a$ ... length of the half major axis
$CD = o_2$ ... minor axis , $|CS| = |DS| = b$ ... length of the half minor axis
$|ES| = |FS| = e$ ... eccentricity
t ... tangent line in M
$a^2 = b^2 + e^2$

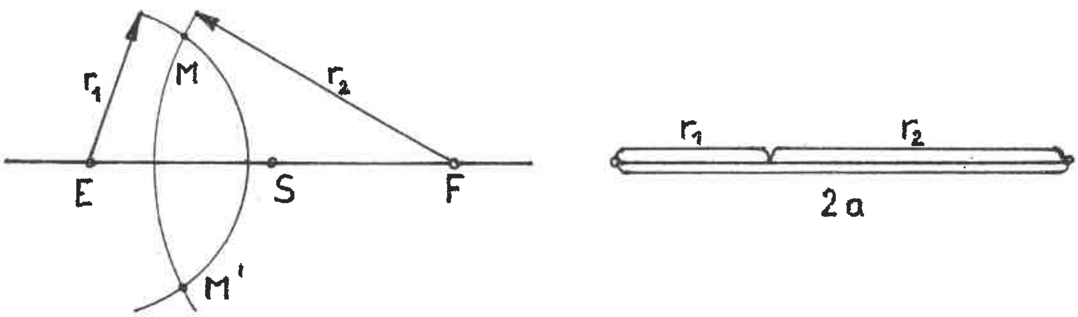
1.1.1 Basic Properties of an Ellipse

- The tangent line passing through the point M (an arbitrary point of an ellipse) bisects the angle $\angle EME'$ between straight lines ME, MF. The normal bisects the adjacent angle $\angle EMF$.
- Let E, E' be symmetrical with respect to the tangent line t. Then $|E'F| = 2a$. It means that all such points E' symmetrical to E with respect to all tangent lines are lying on the circle $k = (F; r = 2a)$. For F' similarly. k is called a **directive circle.**
- Let K be an intersection of tangent line t with a straight line EE', ($EE' \perp t$). Then $|KS| = a$. It means that all such point K are lying on the circle $l = (S; r = a)$. l is called a **vertex circle.**
- Let E, E' be symmetrical with respect to the tangent line t. Then straight line $E'F$ intersects the tangent line t in T, where T is its point of contact with the ellipse.

1.1.2 Constructions of an Ellipse

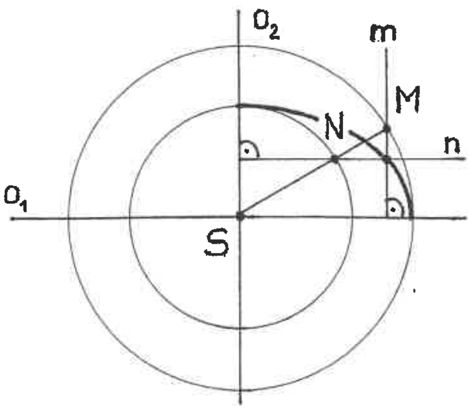
If we know the foci E and F of the ellipse and the real numer $2a$, $2a > |EF|$, we can construct the curve **by means of the definition**, see Fig. 1.2.

Fig. 1.2



One of very simple construction is the **triangle construction**. Draw circles on the major and minor axes using them as diameters and draw any diagonal MN passing through the centre S. From point M draw the line m parallel to the minor axis and from point N the line n parallel to the major axis. The points of intersection of lines m and n are points on the ellipse. See Fig.1.3.

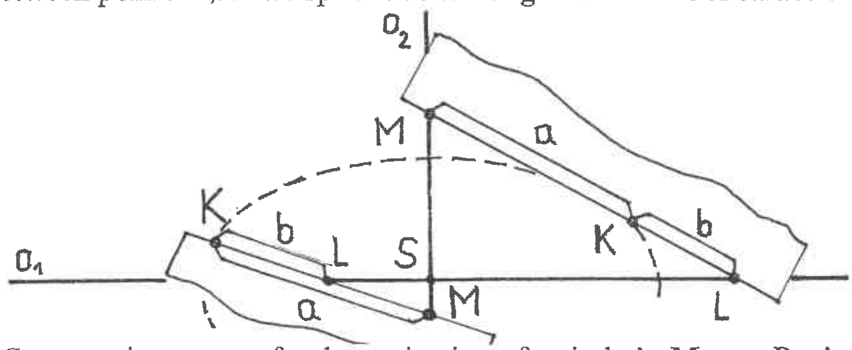
Fig.1.3



An ellipse can be quickly drawn by means of a trammel when the length of the major and minor axes are known. Let o_1, o_2 be axes (lines) of an ellipse and a, b be length of half major and half minor axes. Let KLM be a triple of points on a line such that $|KL| = b$ and

$|KM| = a$. If point L is situated on line o_1 and point M is situated on line o_2 then point K is on the ellipse with the major axis o_1 and the minor axis o_2. If point K is not located between points L, M then the construction is called **Short Trammel Construction**, if point K is situated between points L, M we speak about **Long Trammel Construction**. See Fig.1.4

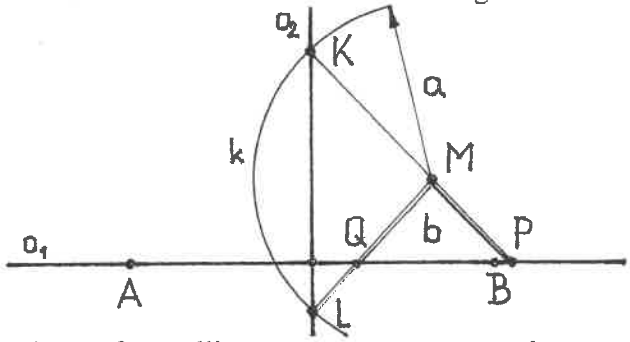
Fig.1.4



The Trammel Construction we use for the projection of a circle **in Monge Projection**. It helps us to circumscribe the length of the half minor axis of the ellipse for which we know the major axis AB and an arbitrary point M.

Example 1.1: Find the minor vertices of the ellipse, when the major vertices and one point of the ellipse are given.
Solution: Let AB be the major vertices of the given ellipse and M its arbitrary point. $AB = o_1$ and $|AB| = 2a$. Then we find S and o_2, $o_2 \perp o_1$ and we draw a circle $k = (M; r = a)$. Let denote by K and L the intersection points of the circle k and the minor axis o_2. Let P be an intersection point of line KM with o_2 and Q be and intersection point of line LM with o_2, then it holds that $|MP| = |MQ| = b$, where b is the length of the half minor axis. Then we can construct the minor vertices. See Fig.1.5

Fig.1.5



Knowing the vertices of an ellipse we can construct the **osculating circles** which osculate the ellipse at vertices. Fig.1.6 shows the construction of the centres of osculating circles. Let A, B, C, D be the vertices of the given ellipse. We construct the rectangle $ASCE$ and the straight line p passing through point E perpendicular to the diagonal AC. The line p intersects the axes of the ellipse at the centres $1, 2$ of the osculating circles. Similar construction without perpendicular line to the diagonal is shown in Fig.1.7.

Fig.1.6

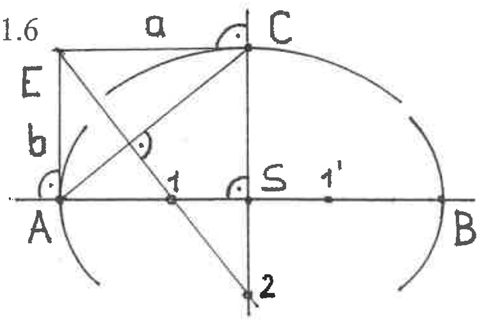
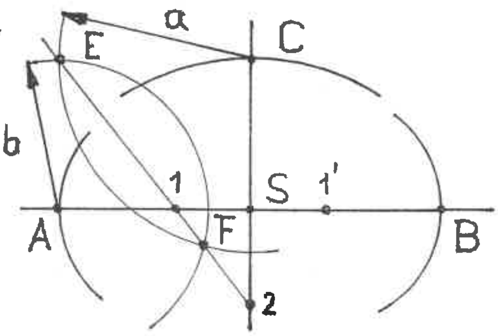


Fig.1.7



Diameter of an ellipse lies on a line passing through the centre S of an ellipse and is bounded by intersection with the ellipse. Two diameters MN, PQ are called **conjugate diameters** if tangent line at end-points of one diameter are parallel with the other one, see Fig.1.8. *1 2 3 4* is a tangent parallelogram, circumscribed to the ellipse. The only one pair of perpendicular diameters of an ellipse consists of the axes of an ellipse.

Fig.1.8

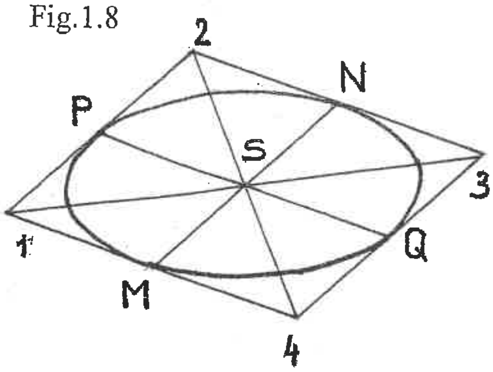
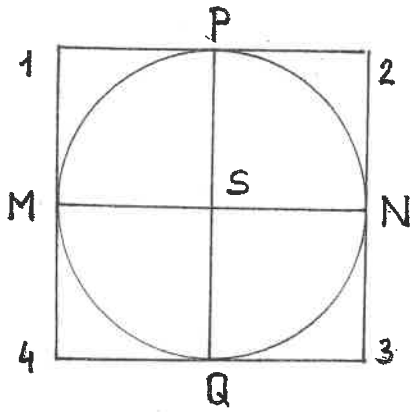


Fig. 1.9

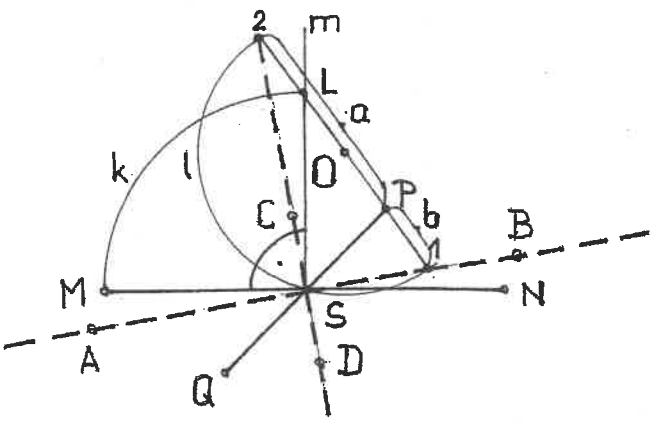


We obtain for a circle: Let MN, PQ be conjugate diameters of a circle. Then $MN \perp PQ$. *1 2 3 4* is a tangent square circumscribed to the circle, see Fig.1.9. It means that conjugate diameters of a circle are orthogonal.

Finding the Axes to an Ellipse with Given Conjugate Diameters (the Rytz's construction).

There are given conjugate diameters MN and PQ . We draw a circle k with S as the centre and MN as the diameter. Draw a line m passing through S perpendicular to MN. Line m intersects the circle k in a point L. Draw the line LP (points L and P lie in the same half-plane determinated by the line MN) and construct the centre O of segment LP. The circle l with the centre O passing through the point S intersects the line LP in points *1* and *2*. Then the major axis lies on the line *1S* and the minor axis lies on the line *2S* and $|2P| = |1L| = a$ and $|1P| = |2L| = b$. See Fig.1.10.

Fig.1.10



Example 1.2: Find the tangent of an ellipse **parallel to a given line**
Solution: The major vertices A, B and the minor vertices C, D of the ellipse and an arbitrary straight line s are given. Find foci E, F ($|CE| = |CF| = a$). Draw a straight line m passing through the focus E perpendicular to s. Then draw the circle $k = (F, r = 2a)$- the directive

circle. The intersections of m with k are E', E''. Draw a straight lines t, t' - points E,E' are symmetrical with respespect to t and E, E'' are symmetrical with respect to t'. Then construct a straight line $E'F$ and find T as its intersection with t. Similarly for E'' and t'. Points T,T' are points of contact of tangent lines t, t' with the ellipse. See Fig.1.11. If E' or E'' is too far from S, we can construct only the accessible one and find the tangent line for it. The other tangent line is symmetrical with respect to S.

This problem we can also solve by finding points K, K' as an intersection of m with the circle $l = (S, r = a)$ - the vertex circle. Tangent lines are $t, K \in t$, and $t', K' \in t'$ parallel to s. See Fig.1.12.

Fig.1.11

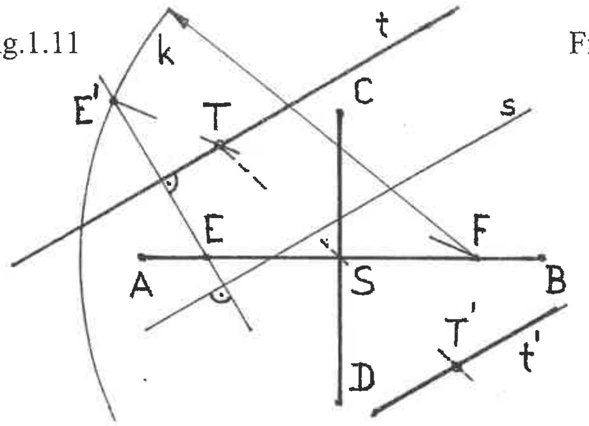
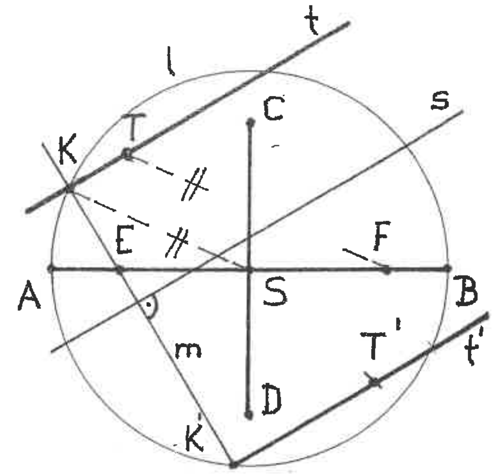


Fig. 1.12



Example 1.3: Finding the tangent of an ellipse, **passing through a given point**.
Solution: An ellipse ε is given and A, B, C, D are its vertices. M is an arbitrary point, $M \notin \varepsilon$. Draw circles $m = (M; r = |ME|)$, $k = (F; r = 2a)$ and find their intersection points E', E''. Then find t as the axis of the symmetry of E, E' and t' as the axis of the symmetry of E, E'' and find T and T' similarly as in Fig.1.11. If E' or E'' is too far from S, construct only one tangent line (for the nearer point) and the other tangent construct using circles $m' = (M; r = |MF|)$ and $k' = (E, r = 2a)$ as shown in Fig.1.13.

Fig.1.14 shows similar construction using K. Construct a vertex circle $l = (S, r = a)$ and then find O as a centre of segment ME. Then draw a circle $m = (O, r = |OM|)$ - Thalet circle. Find K, K' as intersection points of l with m. Tangent lines are $t = KM$ and $t' = K'M$.

Fig.1.13

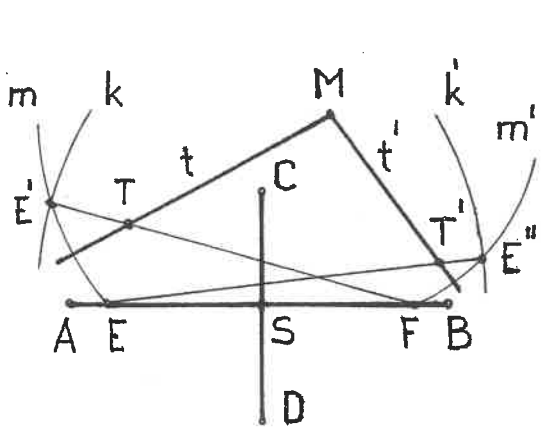
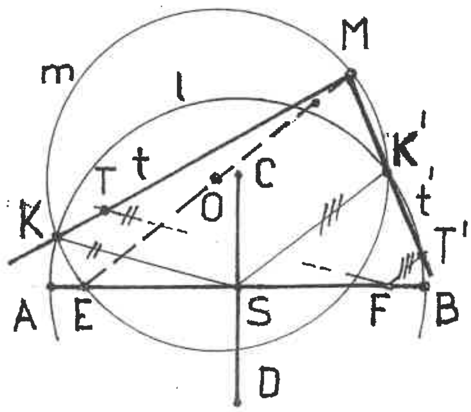


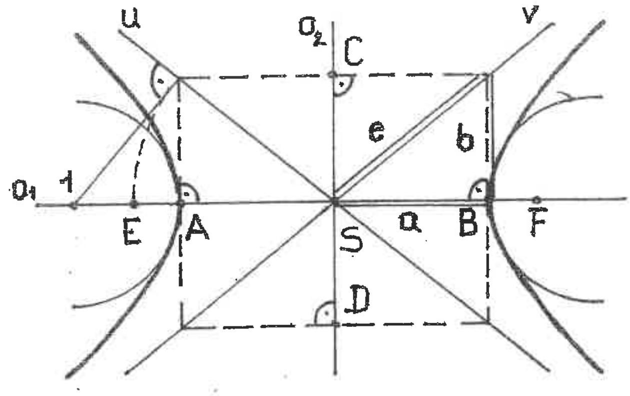
Fig.1.14



1.2 HYPERBOLA

A hyperbola is the set of all points M in the plane whose distances from two fixed points E, F, $E \neq F$ (foci of a hyperbola) have a constant difference $2a, 2a < |EF|$. See Fig. 1.20.

Fig. 1.20



E, F ... focus

A, B ... major vertex, C, D ... minor vertex, S ... centre

u, v ... asymptote

$AB = o_1$ ... major axis, $|AS| = |BS| = a$ ... length of the half major axis

$CD = o_2$ ... minor axis , $|CS| = |DS| = b$ ... length of the half minor axis,

$|ES| = |FS| = e$ ... eccentricity, 1 ... centre of the osculating circle

$e^2 = a^2 + b^2$

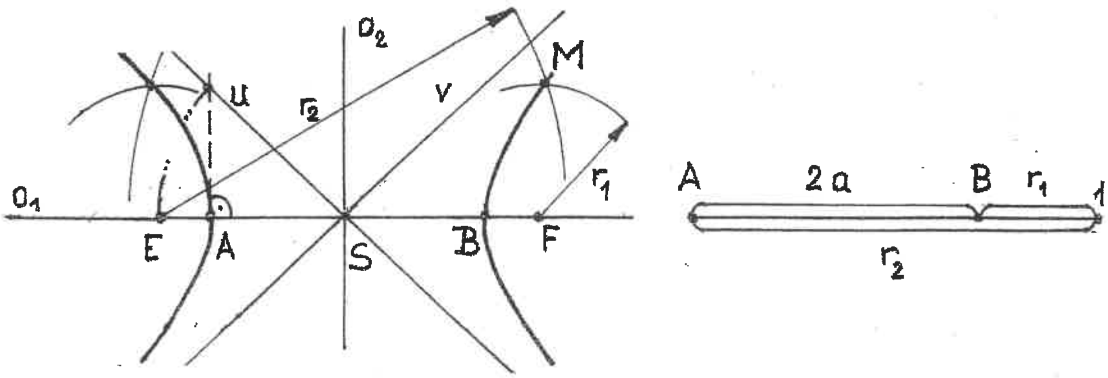
Basic properties of a hyperbola are similar as properties of an ellipse.

There are two important lines for the hyperbola whose are called **asymptotes**.

The asymptotes are the tangent lines of the hyperbola passing through its centre and **their points of contact are at the points at infinity** (see page 11) of the hyperbola.

The construction of a hyperbola by means of the definition is shown in Fig. 1.21. The foci *E,F* and the real number $2a, 2a < |EF|$ are given.

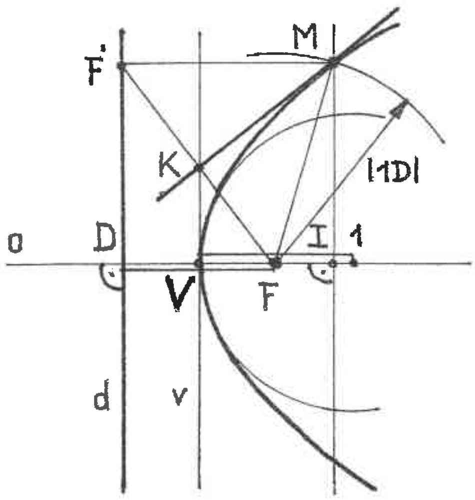
Fig. 1.21



1.3 PARABOLA

A parabola is the set of all points M in the plane that are equidistant from a given fixed point F (focus) and a fixed straight line d (directrix), $F \notin d$. See Fig. 1.30.

Fig. 1.30



d ... directrix
F ... focus
$|dF| = |DF| = p$ ... parameter
V ... vertex
v ... vertex tangent line
1 ... centre of the osculating circle
$|1V| = p$

The construction by means of the definition is shown in Fig. 1.30.

1.3.1 Basic Properties of a Parabola

- The tangent line passing through the point M (an arbitrary point of an ellipse) bisects the angle between straight lines $MF, MF', (MF' \perp d)$. The normal bisects the adjacent angle.
- Points symetrical to F with respect to all tangent lines are lying on **directrix**.
- Let K be an intersection of tangent line t with a straight line FF', ($FF' \perp t$). All such point K are lying on a straight line v, $v \parallel d, V \in v$ - **vertex tangent line**.

2. PROJECTIONS

Both engineering and technical graphics depend on projection methods which were developed to represent 3-dimensional objects on 2-dimensional media (sheet of paper, screen of computer,...). The two projections mostly used are parallel projection and central projection. Projection theory are based on two variables: **line of sight** (projector) and **projection plane.** See Fig. 2.1, Fig. 2.2.

Fig. 2.1

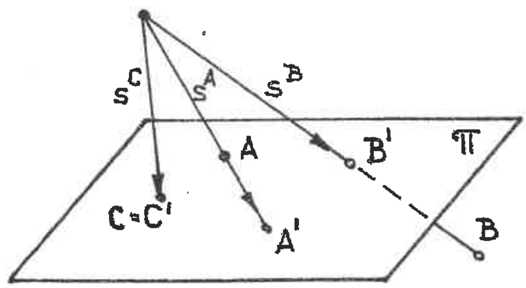
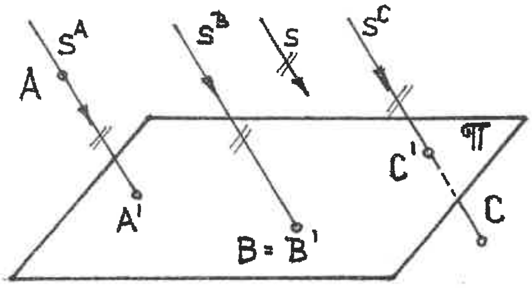


Fig. 2.2



π ... projection plane
s ... direction line of the parallel projection, s $\parallel \pi$
S ... centre of the central projection, $S \notin \pi$
s^A, s^B ... lines of sight (projectors)
$s \perp \pi$ ... **orthogonal (orthographic) projection**
$s \not\perp \pi$ ... skew (oblique) projection

Line of sight (projectors) is an imaginary ray of light between an observer´s eye and an object. In perspective (central) projection all lines of sight start at one point S (the eye), in parallel projection all line of sight are parallel.

Plane of projection π is an imaginary plane upon which the image created by lines of sight is projected. The **view** (image) is produced by connecting points where lines of sight pierce the projection plane π. It means that a 3-D object is transformed into 2-D representation, which is called projection.

Engineering graphics uses different projection methods. One group of such projection methods is based on parallel projection (Monge projection, axonometry, oblique projection,...). Linear perspective is a projection method based on central (perspective) projection. Here we shall use only a parallel projection.

Points at infinity

Let´s assume that a pair $(S;\pi)$, $S \notin \pi$ determines a central projection in the Euclidean space E^3. Within projecting points we find out, that projectors of some points U are parallel to π ($|U\pi| = |S\pi|$), and then we are not able to find their intersections with projection plane π. To overcome this difficulty, let´s assume the existence of a physically fictional "point U_∞', " the projection of U in π. This new object is called **a point at infinity**. The intersection of two parallel lines p and q may be thus expressed as $p \cap q = U_\infty$, and we say that **parallel lines have a common point at infinity**. Exactly one point at infinity lies on any ordinary line and one line at infinity lies on any plane.

2.1 BASIC PROPERTIES OF A PARALLEL PROJECTION

- **Projection of a point** A is a point A', $A' \in \pi$, $AA' \parallel$ s.
- **Projection of a straight line** b, $b \parallel s$ is a straight line b', which we obtained as intersection of the projecting plane β of the line b with the projection plane. Projection of a projecting line c, ($c \parallel s$) is the point C'. See Fig. 2.3.
- **Projection of a plane** ρ not parallel to s is the whole projection plane π. View of a projecting plane δ, ($\delta \parallel$ s) is a straight line δ', $\delta' \equiv p^{\delta}$. See Fig. 2.4.

Fig. 2.3

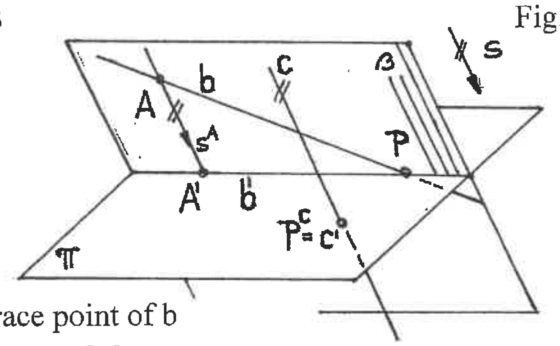
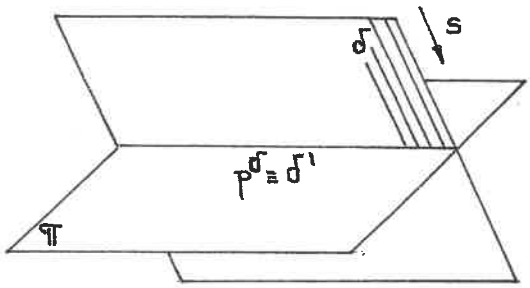


Fig. 2.4



P ... trace point of b

p^{δ} ... trace of δ

- **Projection of a pair of parallel** non-projecting straight lines is a pair of parallel straight lines
- **Projection of a pair of parallel equal segments** lying on the non projecting lines is a pair of parallel equal segments.

Definition

Straight lines parallel to projection plane π are called **principal lines**. Planes parallel to projection plane π are called **principal planes**.

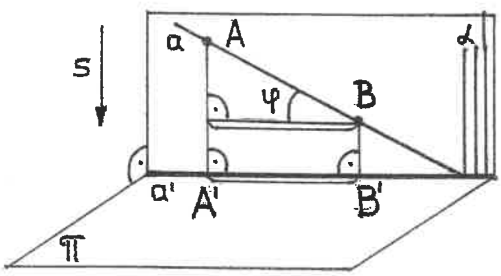
2.2 PROPERTIES OF AN ORTHOGONAL PROJECTION

- **Orthogonal projection shortens segments.** It means that legth of the orthogonal projection $A'B'$ of a segment AB is less or equal to the length of AB. We have

$$|A'B'| = |AB| \cos \varphi ,$$

where φ is the slope angle of the straight line AB according to the plane π. See Fig. 2.5.

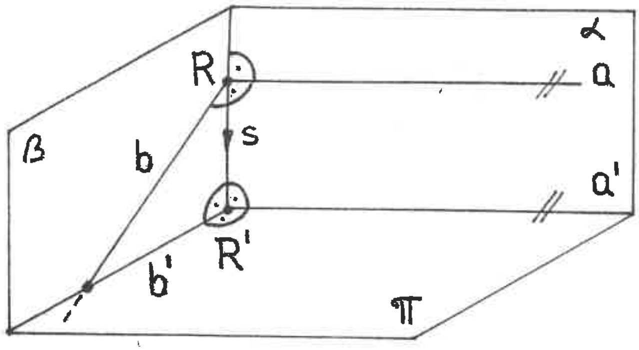
Fig. 2.5



- **Projection of a right angle**

The right angle between non-projecting straight lines a , b is preserved by orthogonal projection if and only if at least one of them is parallel to the plane π . Fig. 2.6.

Fig. 2.6.



α ... projecting plane of a
β ... projecting plane of b
s ... projector of R

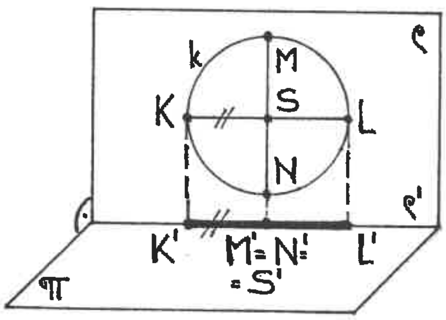
See Fig. 2.6. Let us consider non-projecting stright lines $a, b, a \perp b$ and let us suppose that a is a principal line. Then $a \perp b \wedge a \perp s \Rightarrow a \perp \beta$, $a \perp \beta \wedge b' \in \beta \Rightarrow a \perp b'$, when $\beta = (b, b')$. Therefore $a \perp b' \wedge a' \parallel a \Rightarrow a' \perp b'$.

- **Projection of a circle**

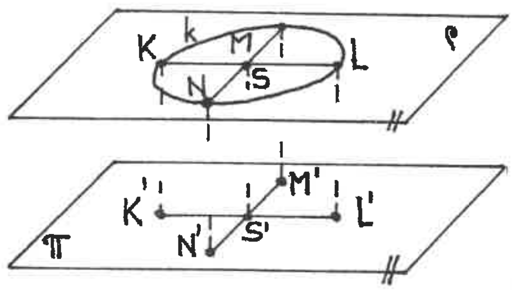
Projection of a circle k is a line segment (k lies on a projecting plane), a circle (k lies on a principal plane) and generally it is an ellipse. See Fig. 2.7.

Fig 2.7

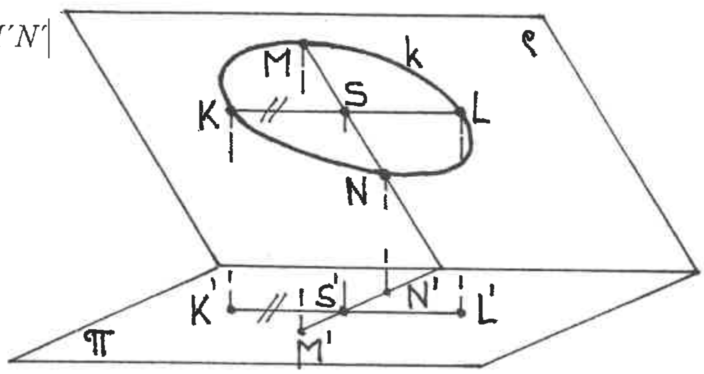
$|KL| = |K'L'|, |M'N'| = 0$



$|KL| = |K'L'|, |MN| = |M'N'|$



$|KL| = |K'L'|, |MN| > |M'N'|$



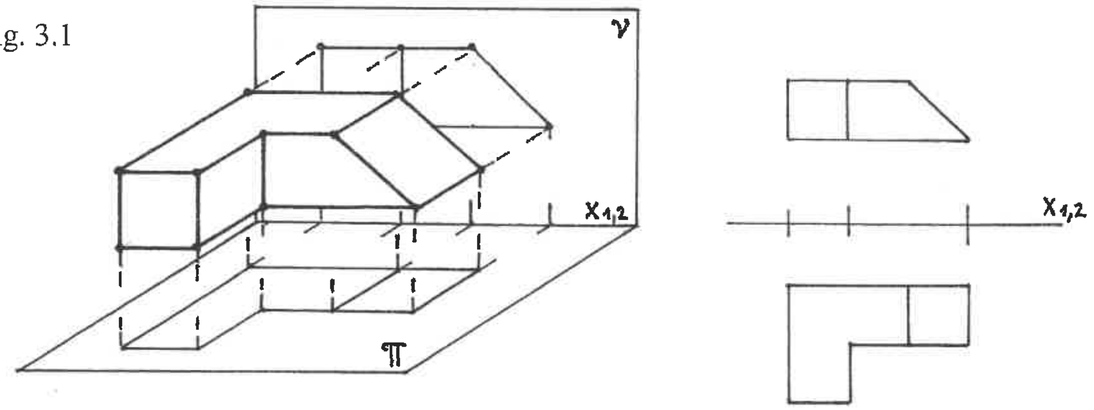
3. MONGE PROJECTION

In the Monge projection we use the orthogonal projections on two mutually perpendicular planes π and ν. The plane π is horizontal and the other one is frontal. The line x of intersection of projecting planes π and ν is called **folding line**, $x \equiv \pi \cap \nu$.

The top view of an object in space is obtained by projecting it downward (or upward) into the horizontal plane π.
The front view of an object in space is obtained by projecting it backward (or forward) into the frontal plane ν.

The horizontal plane is then rotated about folding line x into the frontal plane ν. Resulting arrangement of view of the object in the plane ν is called adjacent (principal) views of the object, see Fig. 3.1. Frontal plane will be identified with the picture plane (screen, sheet of paper, blackboard).

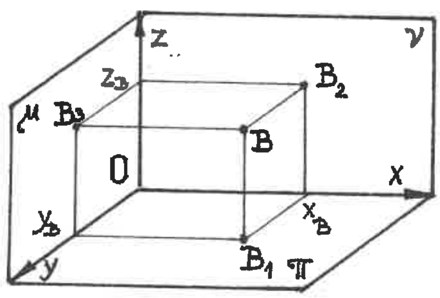
Fig. 3.1



3.1 CARTESIAN SYSTEM OF COORDINATES IN MONGE PROJECTION

We consider three mutually perpendicular planes: the horizontal plane π, the frontal plane ν and the profile plane μ, which is perpendicular to the folding line x, see Fig. 3.2. Point B is orthogonally projected on the planes π, ν, μ and we get its top view B_1, front view B_2 and the side view B_3, see Fig. 3.2.

Fig. 3.2



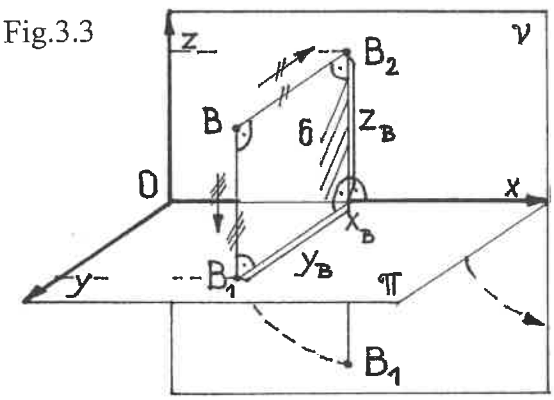
Cartesian system of coordinates in the Monge projection is determined by coordinate planes π, ν, μ. Axes are $x = \pi \cap \nu$, $y = \pi \cap \mu$, $z = \mu \cap \nu$ and the origin O is the common point of π, ν, μ. To measure coordinates, we choose a unit of measurement. Every point B has three coordinates x_B, y_B, z_B in the coordinate system.

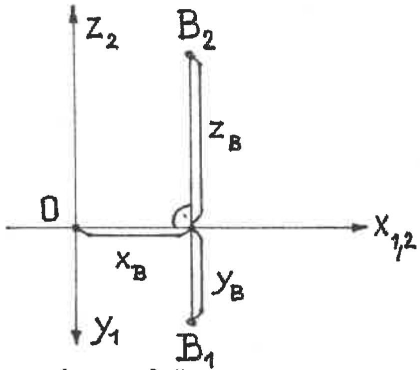
x_B is the oriented distance of B from the profile place μ,
y_B is the oriented distance of B from the frontal place ν,
z_B is the oriented distance of B from the horizontal place π, see Fig. 3.2.

3.2 ADJACENT VIEWS OF A POINT

a) The line connecting adjacent views B_1, B_2, ($B_1 \neq B_2$), is perpendicular to the folding line x.
b) The correspondence between points in the space and their adjacent views is a one to one correspondence.
c) Projecting lines of the point B, ($B \notin \pi, B \notin \nu$), determine a plane σ, which is perpendicular to the folding line x, see Fig. 3.3.

Fig.3.3



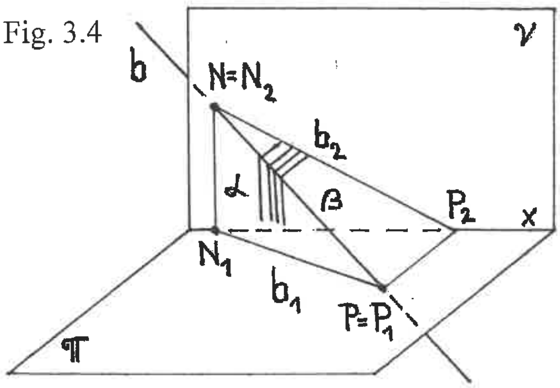


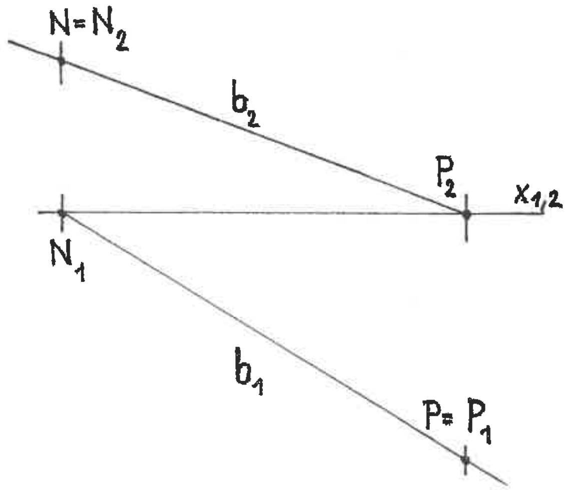
Adjacent views of B

3.3 ADJACENT VIEWS OF A STRAIGHT LINE

$P = b \cap \pi$ is **the horizontal trace point**, $N = b \cap \nu$ is **the frontal trace point** of the straight line b.
α, β are projecting planes of b, $\alpha \perp \pi$, $\beta \perp \nu$, ($b \subset \alpha \wedge b \subset \beta$). The adjacent views of a line b ($b \not\perp \pi, b \not\perp \nu$) consist of a pair of lines, the top view b_1 and the front view b_2. See Fig.3.4.

Fig. 3.4





Adjacent views of b

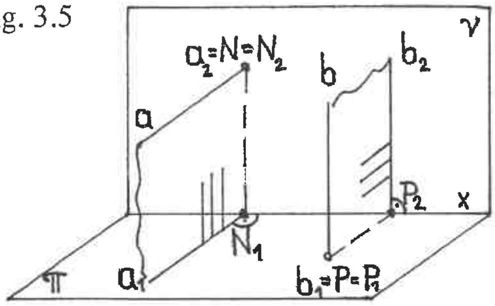
3.4 SPECIAL POSITIONS OF STRAIGHT LINES

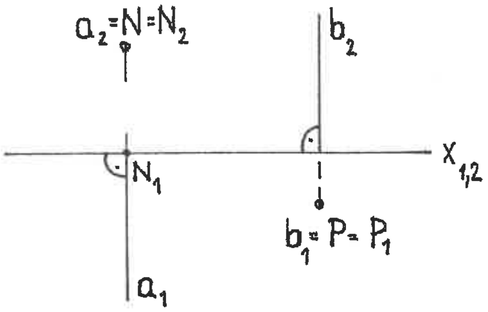
See Fig. 3.5 – 3.9.

3.4.1 Projecting Lines

The front view of the projecting line $a, a \perp \nu$ is the point a_2, the top view of the projecting line $b, b \perp \pi$ is the point b_1. See Fig. 3.5.

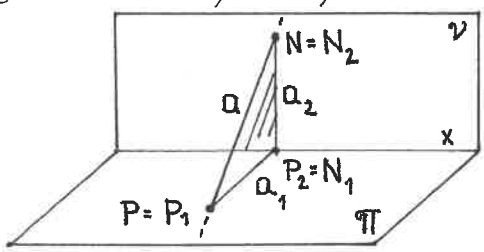
Fig. 3.5

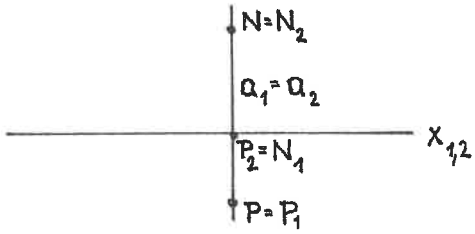




Adjacent views

Fig. 3.6. $a \perp x \wedge a \not\perp \pi \wedge a \not\perp \nu$



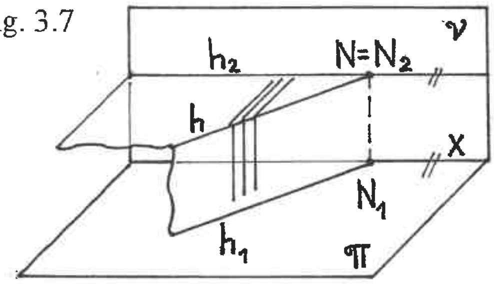


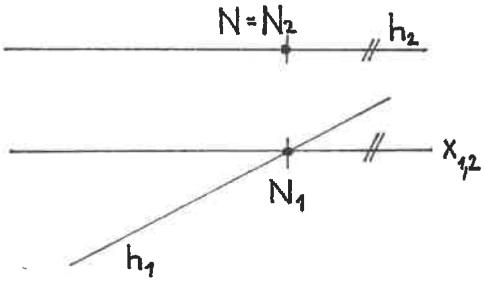
Adjacent views

3.4.2 Principal Lines (lines parallel to a projection plane)

See Fig. 3.7, 3.8.

Fig. 3.7

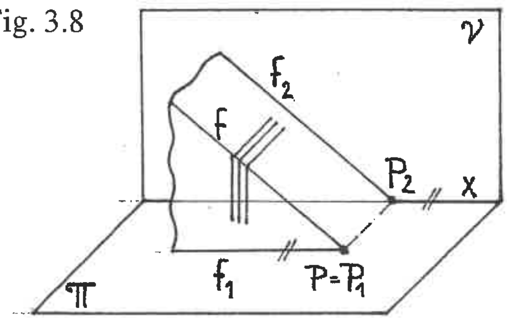


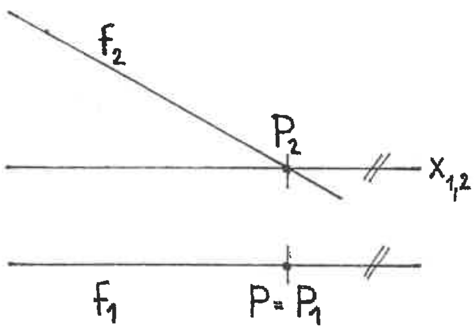


Adjacent views

$h \parallel \pi$ ... horizontal principal line ($h_2 \parallel x_{1,2}$)

Fig. 3.8

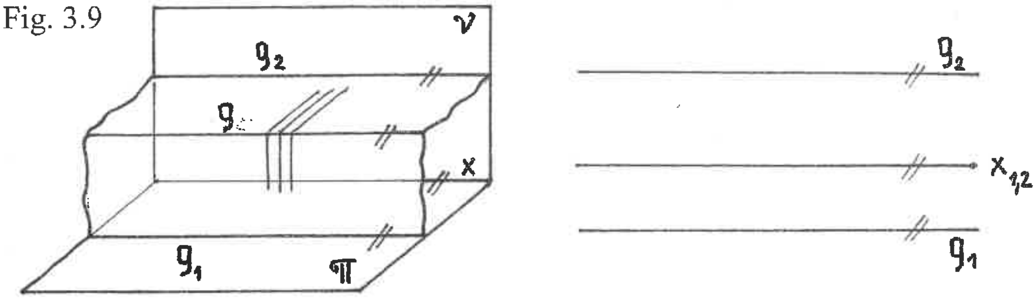




Adjacent views

$f \parallel \nu$ ... frontal principal line ($f_1 \parallel x_{1,2}$)

Fig. 3.9



Adjacent views

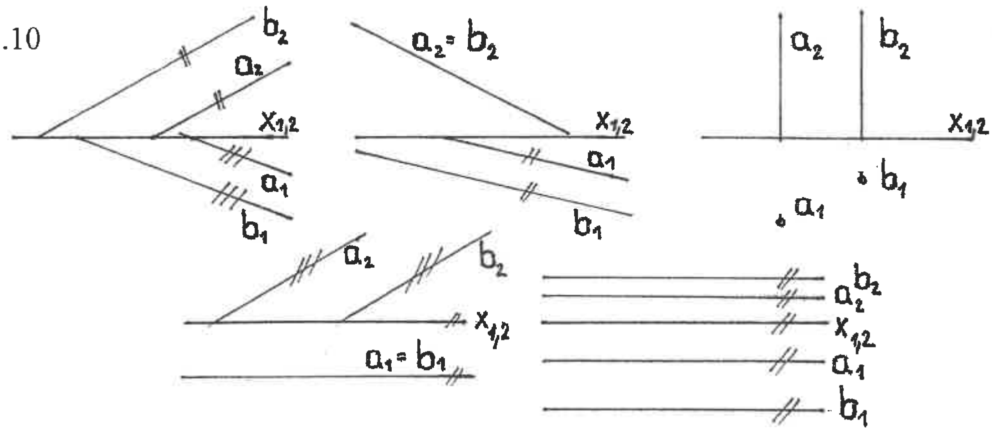
$g \| x$ ... parallel to the folding line x

3.5 A PAIR OF STRAIGHT LINES

3.5.1 Parallels

Several pairs of different parallel lines are presented in Fig. 3.10. The top views of parallel lines are parallel as well as the front views and one pair of views can be incident.

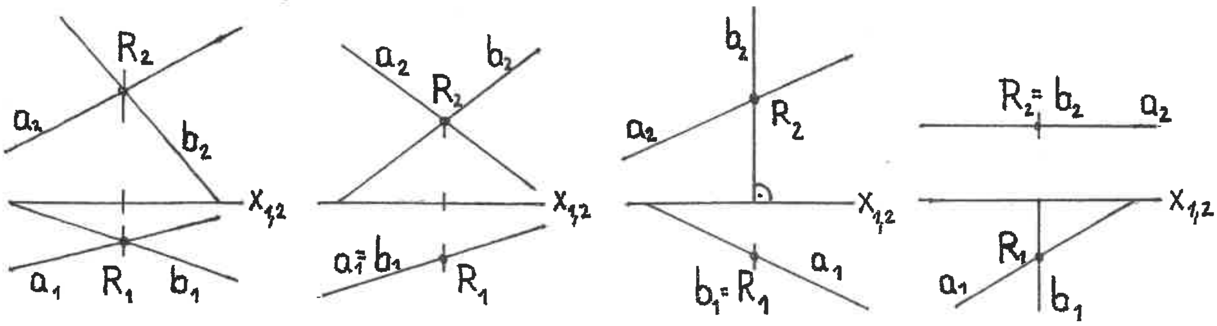
Fig. 3.10



3.5.2 Intersecting Lines

A pair of any intersecting lines has the only one common point. Let us denote R the common point of straight lines a, b. As shown in 3.2 the line connecting the adjacent views of any point R is perpendicular to the folding line x. See Fig. 3.11. It means that both top views and front views of a and b have common points R_1, R_2 and the line connecting the piercing points of the views must be perpendicular to x.

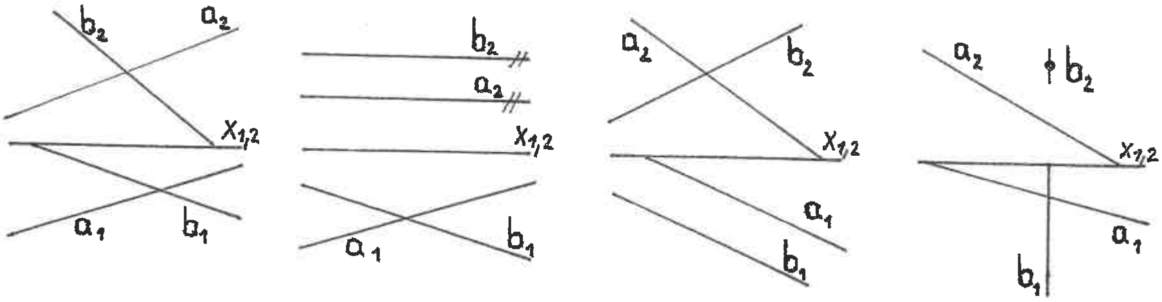
Fig. 3.11



3.5.3 Skew Lines

Skew lines are nonparallel lines that doesn´t have any common point. The adjacent views of the skew lines may have some common points but it is not a pair of adjacent views of a point, it means that the line connecting the piercing points of views (if both exists) is not perpendicular to x. See Fig. 3.12.

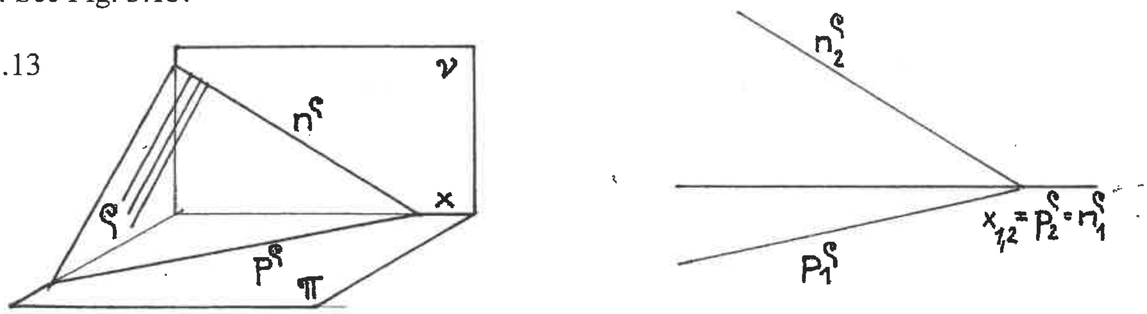
Fig. 3.12



3.6 ADJACENT VIEWS OF A PLANE

Orthogonal views of a nonprojecting plane ρ are projection planes. $p^{\rho} = \rho \cap \pi$ is the **horizontal trace** of the plane ρ, $n^{\rho} = \rho \cap \nu$ is the **frontal trace** of the plane ρ. In the Monge projection a plane is given by adjacent views of its determining elements (point, lines). See Fig. 3.13.

Fig. 3.13



The top view of a projecting plane $\alpha, \alpha \perp \pi$ is a line – its horizontal trace p^{α}

The front view of a projecting plane $\beta, \beta \perp \nu$ is a line – its frontal trace n^{β}. See Fig. 3.14, Fig.3.15.

Fig. 3.14

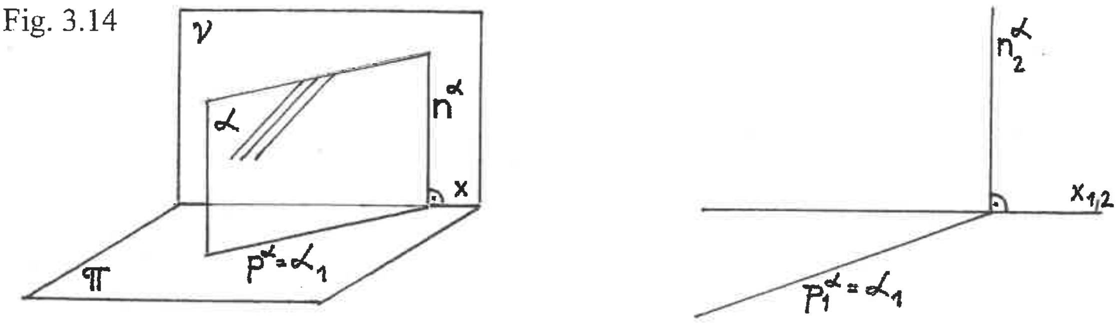
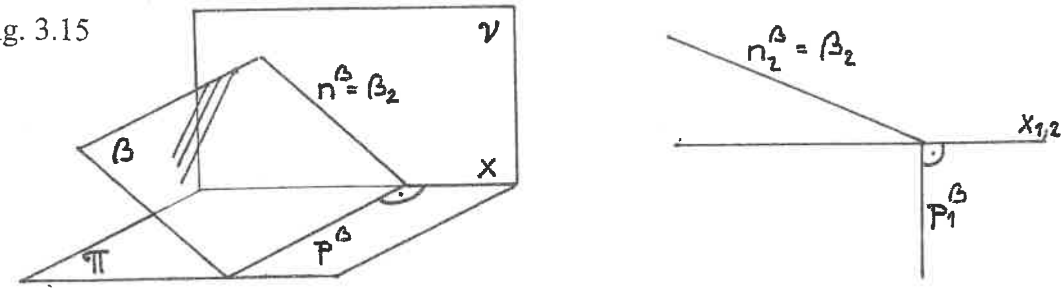


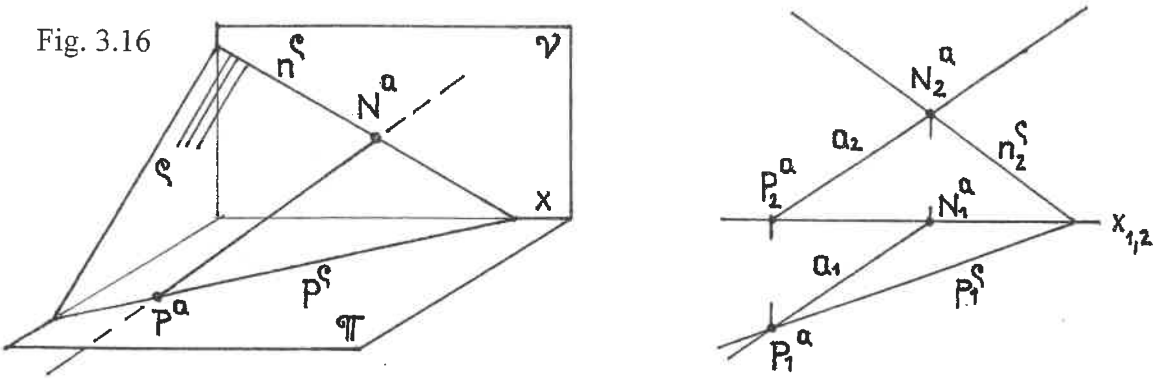
Fig. 3.15



3.7 A STRAIGHT LINE IN A PLANE

If the straight line a lies in the plane ρ and exist their trace points P^a, N^a and p^ρ, n^ρ, then the trace points $P^a \in p^\rho \wedge N^a \in n^\rho$. See Fig. 3.16 – 3.18.

Fig. 3.16



SPECIAL LINES IN A PLANE

3.7.1 Principal Lines in a Plane

Adjacent views of the principal lines of the plane ρ.

Fig. 3.17 Horizontal principal line

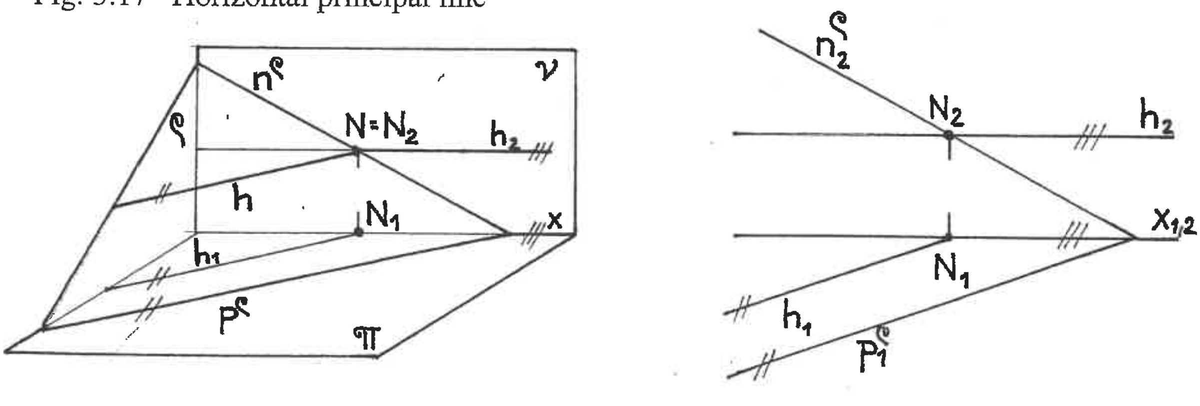
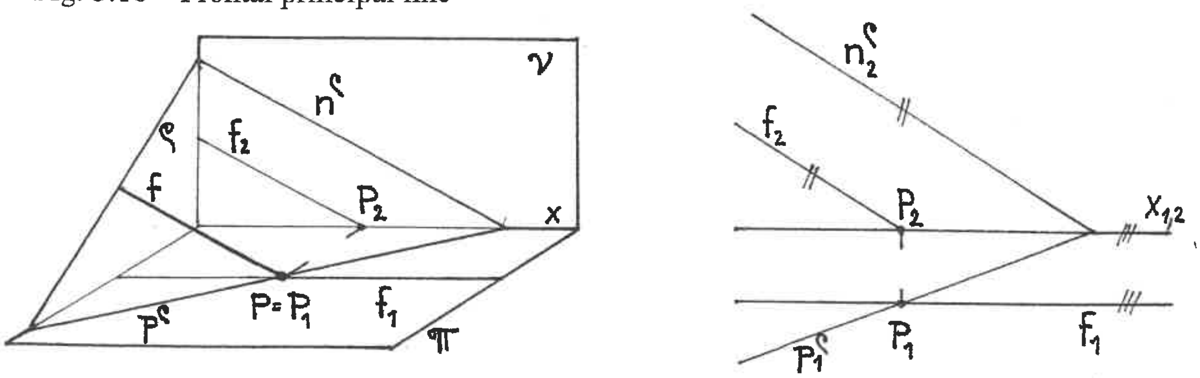


Fig. 3.18 Frontal principal line

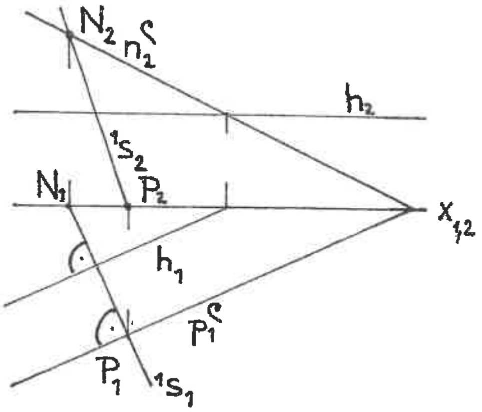


3.7.2 Steepest Lines

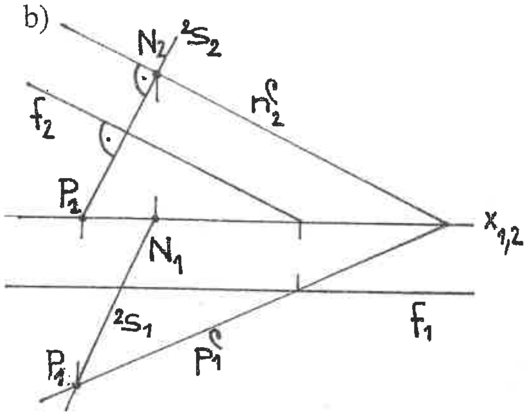
The slope angle of the plane is the angle α that the plane σ makes with the projection plane π. A line ^1s in the plane having the same slope angle as the plane makes is called a **steepest line** in the plane. The steepest line ^1s and the projection plane π make the greatest slope angle. The steepest line ^1s in plane is always perpendicular to horizontal principal line *h*, then top view $^1s_1 \perp h_1$. (The situation is similar for steepest line ^2s, the projection plane ν and the frontal principal line f and its front view f_2). See Fig. 3.19a, Fig.3.19b.

Fig. 3.19

a)



b)



3.8 A POINT IN A PLANE

For construction of adjacent views of a point B lying in a plane ρ we can use any straight line b, $B \in b$ lying in a plane ρ. Principal lines are often used for this construction.

3.1 Example: Construct adjacent views of the points $B \in \rho$ and $C \in \rho$.

Solution: The top views B_1 , C_1 are given. We construct missing views by using horizontal principal lines, see Fig. 3.20a), or by using frontal principal lines, see Fig.3.20 b). If front views B_2, C_2 are given, we can use similar construction.

Fig. 3.20 a)

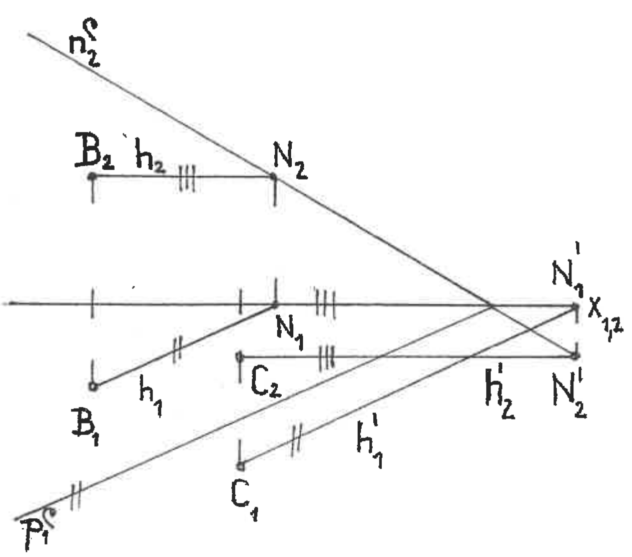
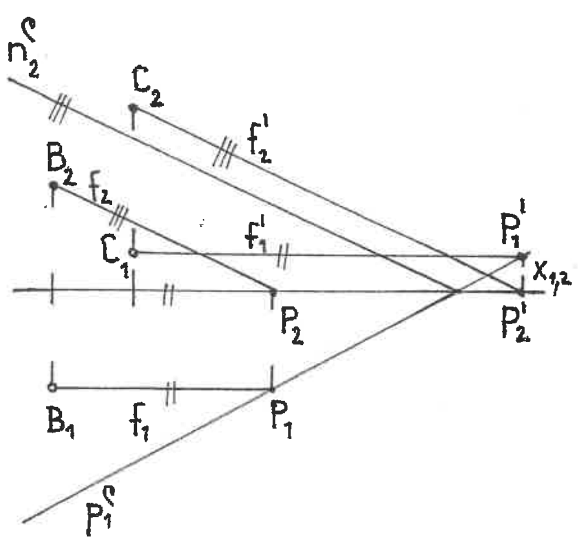


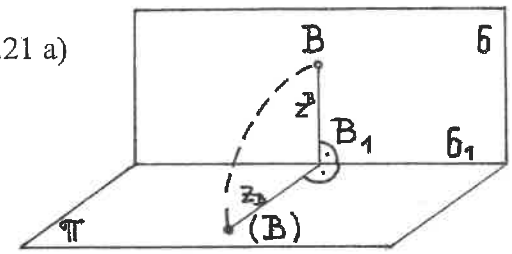
Fig. 3.20 b)



3.9 ROTATION OF A PROJECTING PLANE INTO A PROJECTION PLANE

See Fig. 3.21. We shall rotate plane $\sigma, (\sigma \perp \pi)$ into the projection plane π (or into a horizontal plane $\alpha \parallel \pi$). The axis of the rotation is σ_1 (or the horizontal principal line $h = \alpha \cap \sigma$). The trajectory of the point B of the rotating plane σ is a circle, which lies in the plane β perpendicular to the axis of rotation , its centre is $B_1 = \sigma_1 \cap \beta$ and has the radius $r = z_B$, see Fig. 3.21 a) (or $r = |z_B - z_\alpha|$), see Fig. 3.22 . Rotation of frontal projecting plane $\omega, \omega \perp \nu$, is similar, see Fig. 3.21 b).

Fig. 3.21 a)



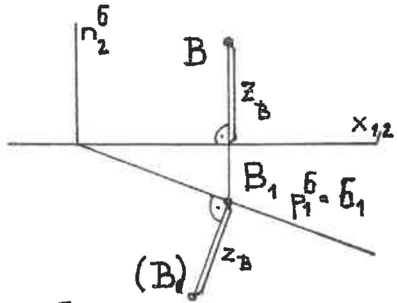


Fig. 3.21 b)

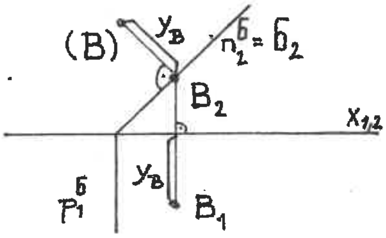
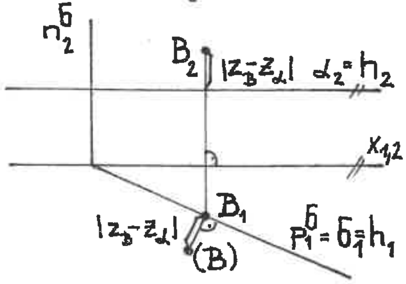


Fig. 3.22



3.2 Example: Construct the true length of segment AB.
Solution: We use the vertical projecting plane $\sigma, \sigma \perp \pi$, passing through AB and rotate it into π (or into $\alpha \parallel \pi$), see Fig. 3.23. $|(A)(B)|$ is the true length of the segment AB.

Fig. 3.23

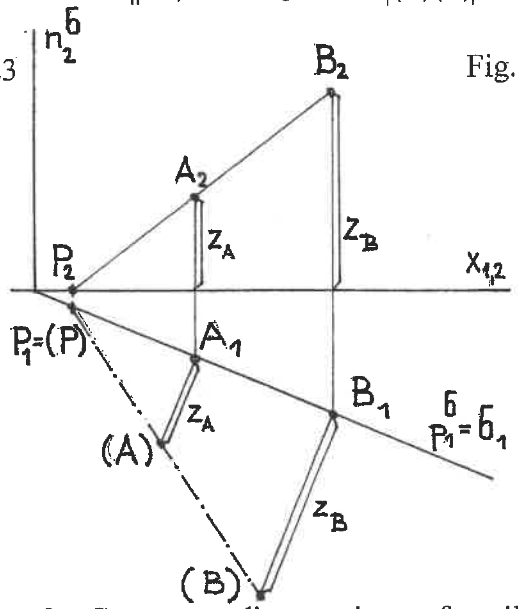
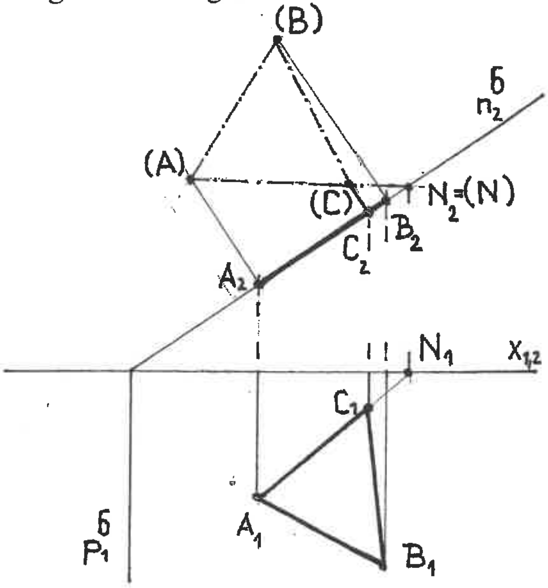


Fig. 3.24



3.3 Example: Construct adjacent views of equilateral triangle ΔABC lying in the projecting plane $\sigma, \sigma \perp \nu$. Side AB of the triangle is given.
Solution: We rotate the projecting plane σ into ν. Now we construct equilateral triangle $\Delta(A)(B)(C)$. Point (C) is rotated back into C, see Fig.3.24, problem has two solutions.

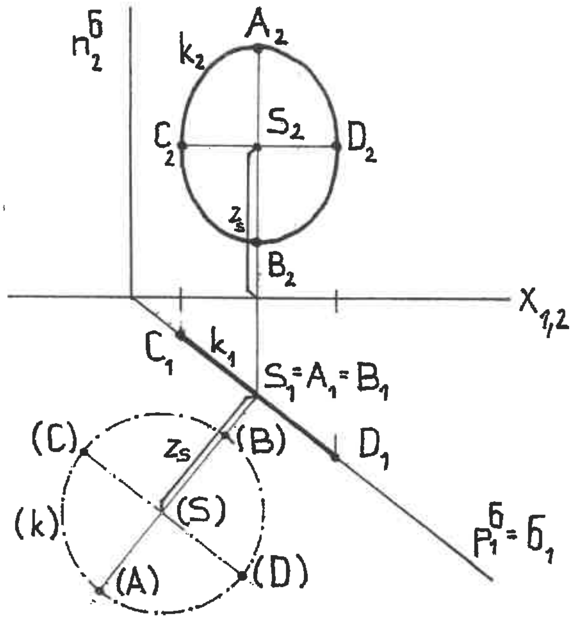
3.4 Example: Construct adjacent views of the circle $k = (S;r)$ lying in the projecting plane $\sigma, \sigma \perp \pi$.

Solution:

1. We rotate vertical projecting plane σ into π.
2. We construct the rotated circle $(k) = ((S);r)$
3. We construct diameters *(A)(B), (C)(D)* of the circle *(k)* and rotate it back.

The top view k_1 of the circle is the segment $C_1D_1, |C_1D_1| = 2r$, because $CD \parallel \pi$. The front view is an ellipse with the major axis A_2B_2 (AB $\parallel \nu$ and then $|AB| = |A_2B_2|$) and the minor axis C_2D_2. We can see easily that the adjacent views of a circle in a projecting plane can be constructed without rotation. See Fig. 3.25.

Fig. 3.25



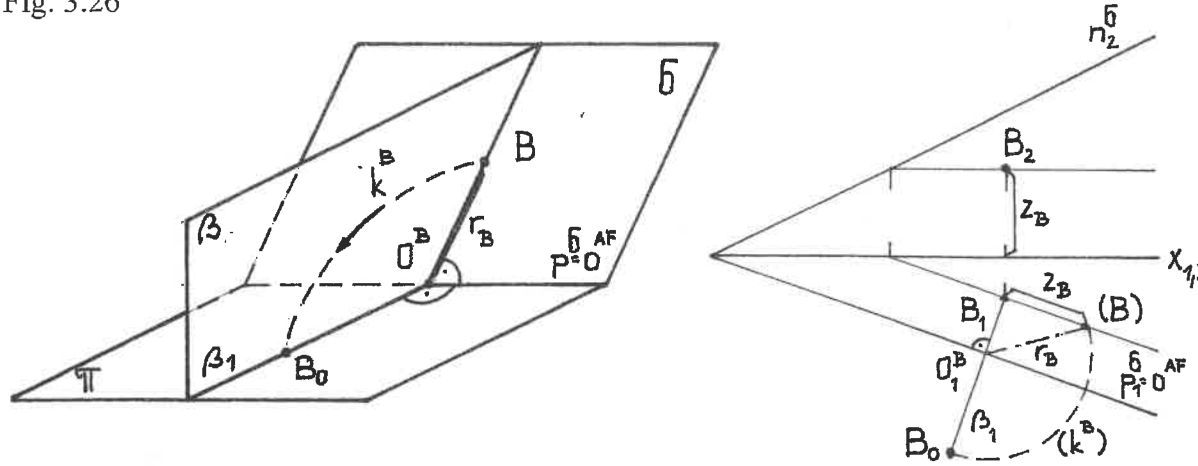
3.10. ROTATION OF AN ARBITRARY PLANE INTO A PROJECTION PLANE

It is obvious that the projection of an object lying in an arbitrary plane has another shape and form. There are several methods how to construct a given plane object from certain conditions or how to construct a true shape of a plane object if we know its adjacent views. One of these methods is the rotation of this plane into a projection plane or into a plane parallel to a projection plane.

Let's rotate an arbitrary plane σ into a projection plane π. The axis of the rotation is $p^\sigma = \sigma \cap \pi$. The trajectory of the point B of the rotating plane σ is a circle k^B. This circle lies in the plane $\beta, \beta \perp p^\sigma$. The top view of β is a straight line β_1 perpendicular to p^σ and the top view of k^B is a segment of it. The centre of k^B is $O^B = \beta \cap p^\sigma$ and its radius is the true length of the segment BO^B. We construct the true length by rotation of the projecting plane β into π and the circle (k^B) intersects the line β_1 at the point B_0. See Fig. 3.26.

The correspondence $B_1 \mapsto B_0$ is **an axial affinity** (see par. 3.10.1) with the axis p_1^β, points B_1 and B_0 are the corresponding points. The correspondence used for rotation of an arbitrary plane is an orthogonal axial affinity ($B_1B_0 \perp o^{AF}$).

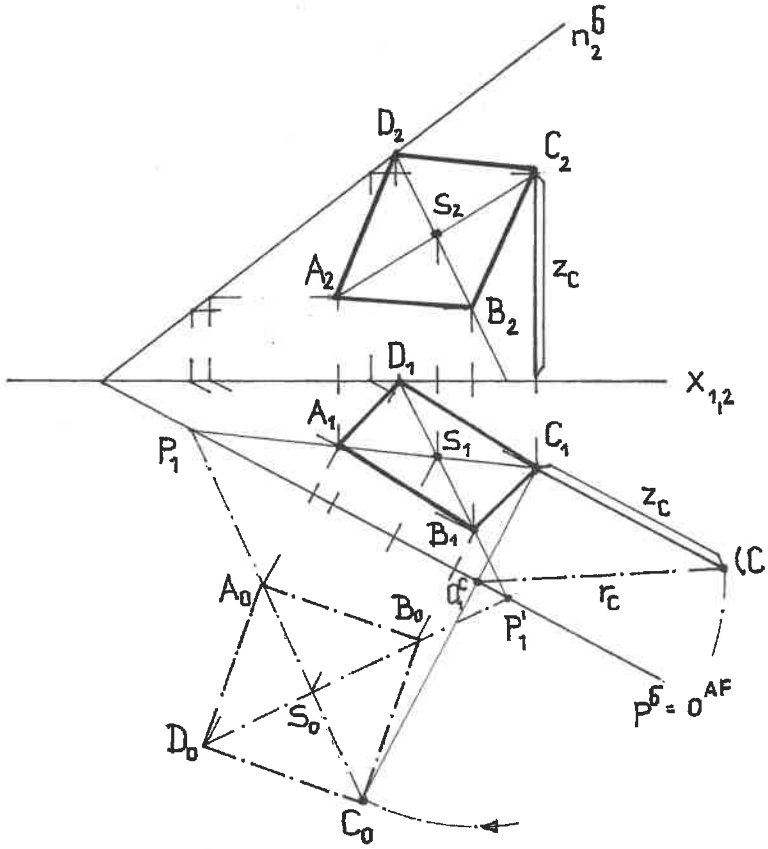
Fig. 3.26



3.5 Example: Construct the adjacent views of the square $ABCD$ lying in the given plane σ. The top views A_1, C_1 of the points A and C are given. see Fig. 3.27.

Solution: We locate front views A_2, C_2. Then we rotate the plane σ about the line p^σ into projection plane π and find C_0 using the last construction (C is more suitable for good precision than B). Then we locate A_0 using the orthogonal axial affinity defined by the axis $p^\sigma = o^{AF}$ and the pair of corresponding points $C_1 \mapsto C_0$. P_1 is the double point of the pair of corresponding lines $A_1C_1 \mapsto A_0C_0$, $A_0A_1 \perp o^{AF}$. We draw the square $A_0B_0C_0D_0$. An inverse procedure gives us D_1: $D_0D_1 \perp o^{AF}$ and we use the double point P'. The projections of a square is a parallelogram in both views. We draw the front view by means of horizontal principal lines of the plane σ.

Fig. 3.27



3.10.1 Axial Affinity

An axial affinity in a plane is the corespondence defined by an axis of affinity (fixes line) o^{AF} and a pair of corresponding points $A \mapsto A'$ (o^{AF} , A, $A'$are lying in one plane).

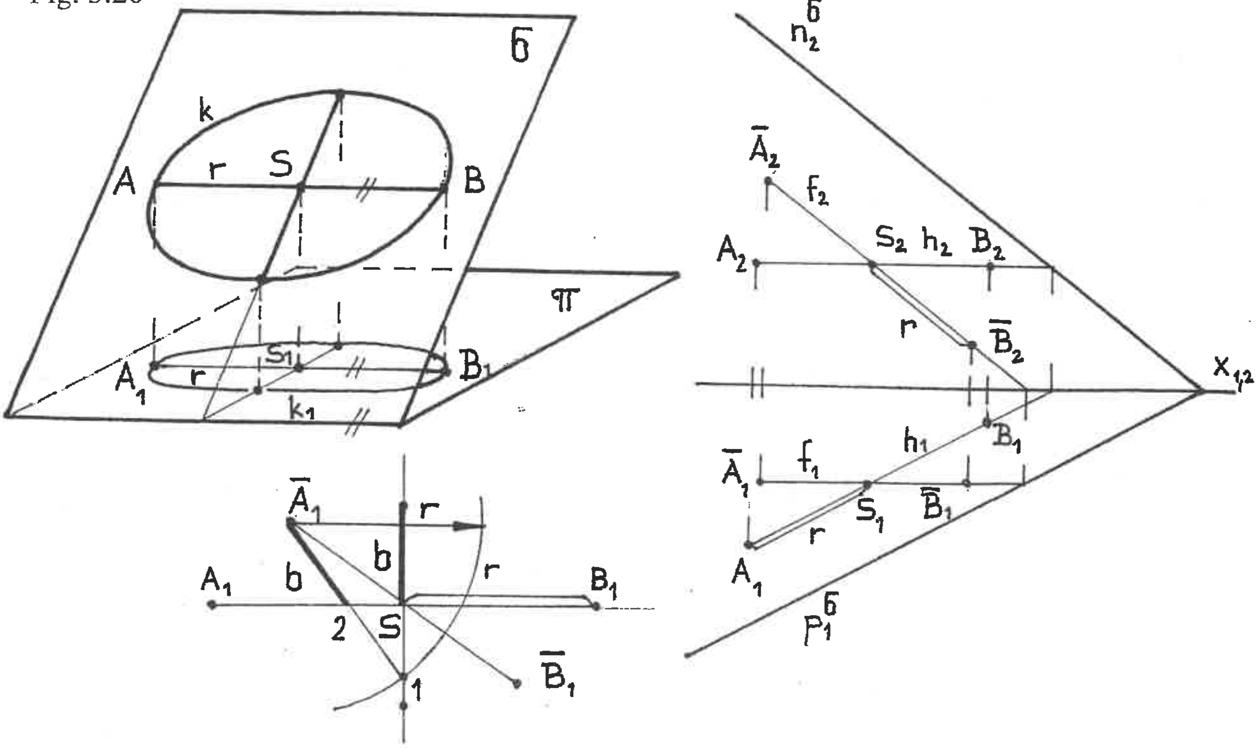
Basic properties of an axial affinity:

- Let B and B' be a pair of corresponding points. Then $BB' \| AA'$.
- Let p and p' be a pair of corresponding straight lines. If p intersects the axis of affinity o^{AF} in the point P, then p' also intersects o^{AF} in P. P is called **double point** (P can be also a point at infinity).

3.11 A CIRCLE IN AN ARBITRARY PLANE

We suppose a given circle k lying in the plane σ. This plane σ is neither parallel to a projection plane (we know that in this case one view of the circle is a circle and the second one is a segment) and nor σ is perpendicular to a projection plane (see par. 3.9). The general situation modelled in Fig. 3.28 shows that the orthogonal projection of such a circle is an ellipse. Its major axis lies on a line parallel to the image plane and contains the centre of the circle. The half major axis is equal to the radius of the circle. In the front view the major axis lies on the front view f_2 of the frontal line f passing through the front view S_2 of the centre S and the vertices are points $\overline{A_2}$ and $\overline{B_2}$, $|S_2\overline{A_2}| = |S_2\overline{B_2}| = r$. Similarly, the major axis of the top view lies on the top view h_1 of the horizontal line h passing through S_1, $|S_1A_1| = |S_1B_1| = r$. The minor axis in each view lies on a line perpendicular to a major axis. This line is the steepest line in the plane σ. Because point A_2 and B_2 lie on the front view of the circle and the point $\overline{A_1}$ and $\overline{B_1}$ on the top view of the circle respectively, the circle can be drawn by means of a **trammel construction** (see Fig. 1.5).

Fig. 3.28



3.12 A LINE PERPENDICULAR TO A PLANE

The basic property of the orthographic projection describes the orthographic projection of a right angle. A line k is perpendicular to a plane σ if and only if it is perpendicular to all lines lying in this plane, so the searched line k is perpendicular to all horizontal and all frontal lines in the given plane. According to par. 2.3 (projection of a right angle) the projections of the right angle $\angle kh$, $\angle kf$ is the right angle $\angle k_1 h_1$ and the right angle $\angle k_2 f_2$. $p^\sigma \parallel h$ and $n^\sigma \parallel f$, then $k_1 \perp p_1^\sigma \wedge k_2 \perp n_2 \sigma$, see Fig. 3.29.

Fig. 3.29

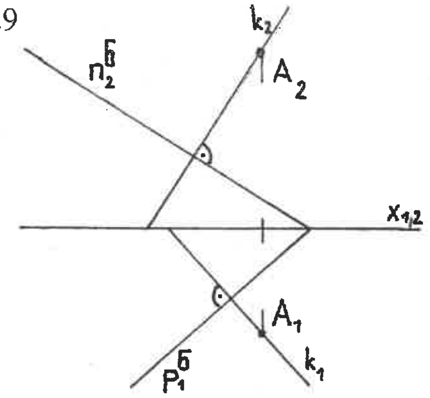
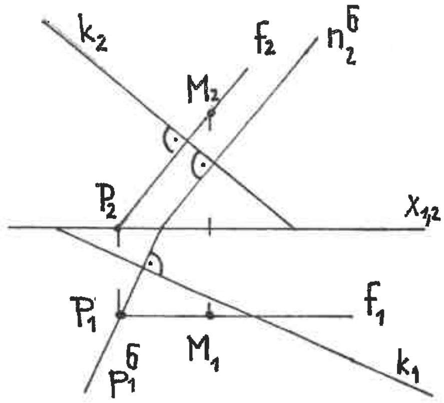


Fig. 3.30

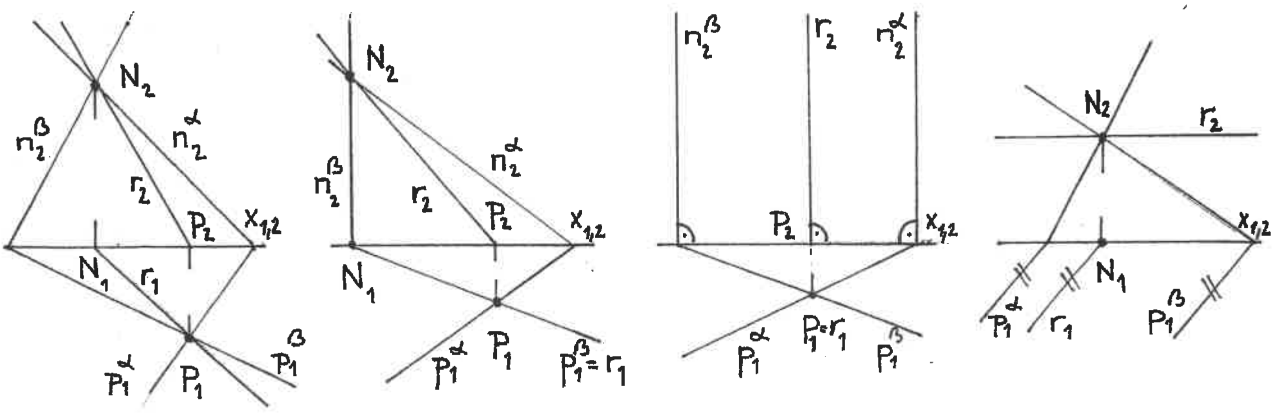


3.6 Example: Draw a plane σ passing through the point M perpendicular to the line k.
Solution: We know that the line k is perpendicular to the plane σ and then $k_1 \perp h_1 \wedge k_2 \perp f_2$. We choose the principal line f passing through the point M and find its trace point P. We construct the trace p^σ passing through the point P, $\left(p_1^\sigma \perp k_1\right)$, and $n^\sigma, \left(n_2^\sigma \perp k_2\right)$, see Fig. 3.30.

3.13 AN INTERSECTION OF TWO PLANES

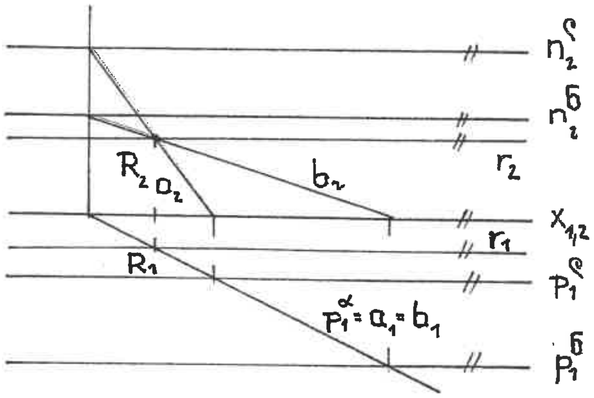
For construction of an intersection line of two given planes we simply find common points of these planes - for example intersection points of the traces, see Fig. 3.31.

Fig. 3.31



3.7 Example: Draw the intersection line r of the two given planes ρ, σ, ($\rho \parallel$ x, $\sigma \parallel$ x).
Solution: We can easily see that the intersection line r is also parallel to x. For the construction we can use any arbitrary plane α. We construct intersection lines a, $a = \alpha \cap \rho$ and b, $b = \alpha \cap \sigma$. The point R, $R = a \cap b$ is a common point of these three planes α, ρ and σ, so now we can draw the straight line r parallel to x, $R \in r$. See Fig. 3.32.

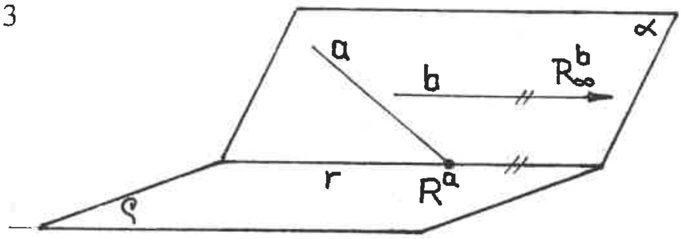
Fig. 3.32



3.14 AN INTERSECTION OF A STRAIGHT LINE AND A PLANE

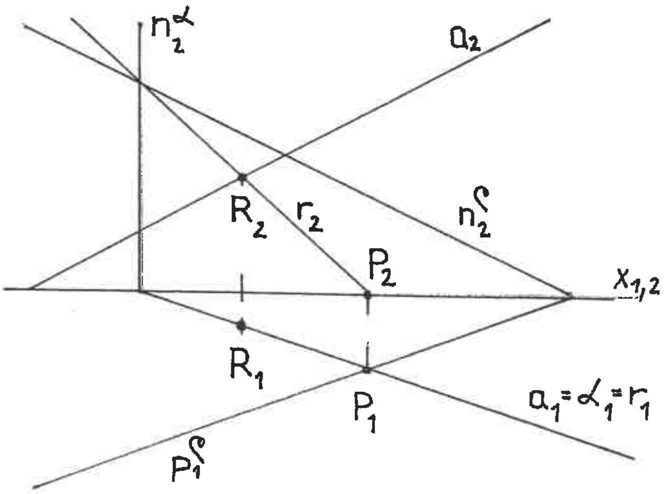
Let us assume two planes α, ρ and their intersection line r. We assume straight lines lying in α. We can say that each of these lines intersects ρ in a point lying in the straight line r (if the line is parallel to ρ, it is also parallel to r, it means that the piercing point is the point at infinity $R_\infty, R_\infty \in r$). See Fig. 3.33.

Fig. 3.33



3.8 Example: Construct the piercing point R between the given plane ρ and the line a.
Solution: First we construct any plane $\alpha, a \subset \alpha$, for easy construction we use α perpendicular to a projection plane. Then we find the intersection line r between planes α and ρ and finally we construct an intersection point R between the straight lines a and r. See Fig. 3.34.

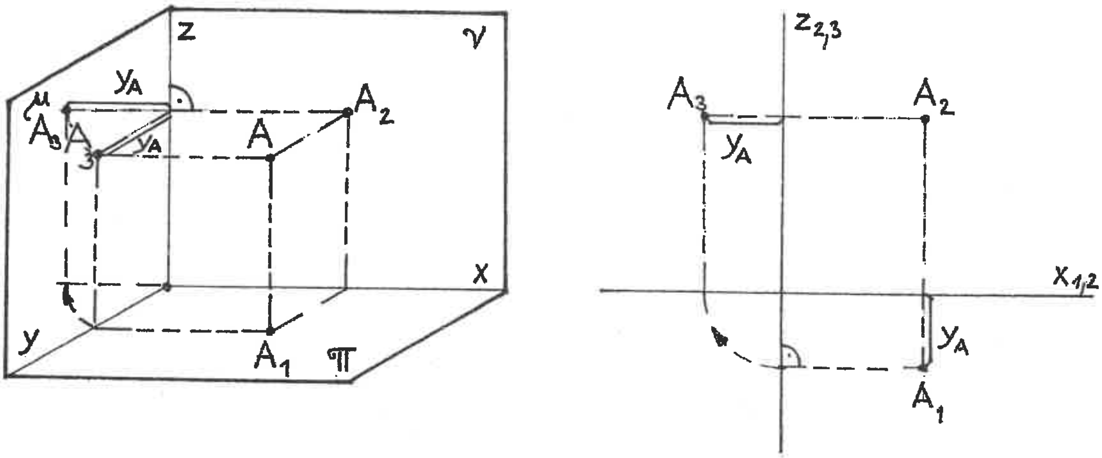
Fig. 3.34



3.15 SIDE (PROFILE) VIEW

In Monge projection we project orthogonally on two coordinate planes, horizontal plane $\pi = (x, y)$ and frontal plane $\nu = (x, z)$. Often a third projection is used, an orthogonal projection onto the plane $\mu = (y, z)$, which is called the side (or profile) plane, see Fig. 3.3. Point A is orthogonally projected onto μ, we obtain the side view A_3 of the point A. Side plane is folded with the frontal plane ν(or with the horizontal plane π), which means that the side plane is rotated about z onto ν. In this way we obtain a new view of the point A, denote it A_3 and call it a side view of A, see Fig. 3.35. We obtain a new pair of adjacent views of the point A: the front view A_2 and the side view A_3, for which $A_2A_3 \perp z$. The z axis is the new folding line and the line connecting adjacent views is perpendicular to it.

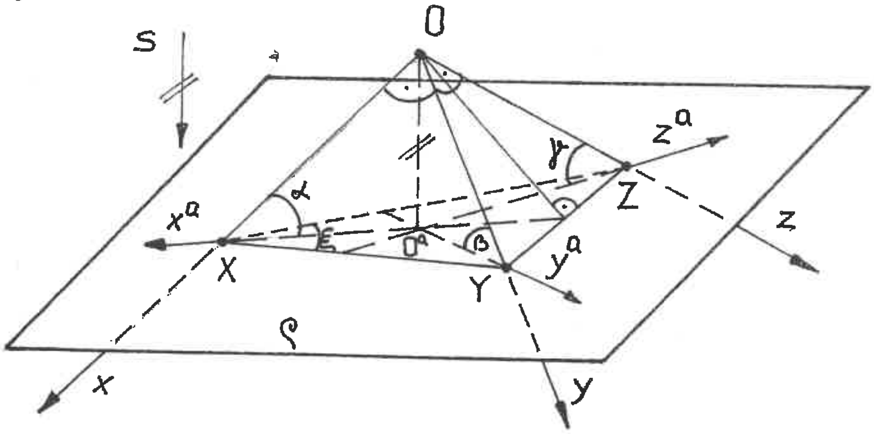
Fig. 3.35



4. ORTHOGONAL AXONOMETRY

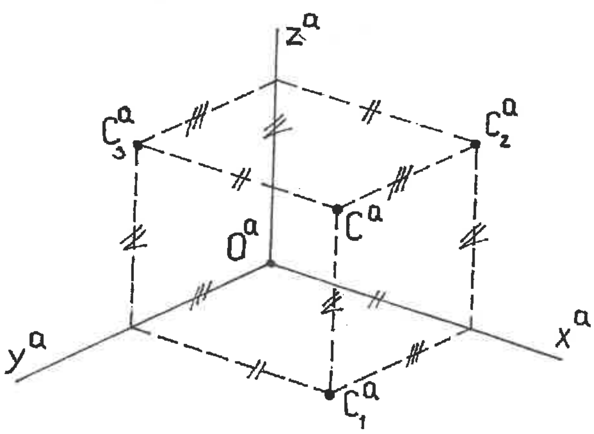
Orthogonal axonometry (further an axonometry) is an orthogonal projection on projection plane (picture plane) ρ, has general position with respect to the Cartesian coordinate system (*O, x, y, z*), see 3.1 and Fig. 3.2, Fig.4.1.

Fig.4.1



Projectors are perpendicular to the picture plane ρ. Any point in the space is represented by a pair of axonometric views: Axonometric view C^a of the point C and axonometric view C_1^a of its top view C_1 on the plane $\pi = (x, y)$, see Fig. 4.2. (Instead of the axonometric view of the top view can be given the axonometric view of the front view or the side view and the other ones can be simply found). Projection defined this way determinates a one to one correspondence between points C in the space and the pairs of points C^a, C_1^a in the picture plane ρ.

Fig. 4.2: Axonometry of the coordinate box of C



All coordinates of points are shortened. Axonometric axes x^a, y^a, z^a are axonometric views of axes *x, y, z*. Vertices of the axonometric triangle ΔXYZ are points of intersection of axes *x,y,z* with the picture plane ρ. Denote $\alpha = \angle x\rho, \beta = \angle y\rho, \gamma = \angle z\rho$.

4.1 PROPERTIES OF ORTHOGONAL AXONOMETRY

1) Axes x^a, y^a, z^a contain altitudes of the axonometric triangle ΔXYZ, see Fig. 4.3.

Proof. From Fig. 4.1 we see that $YZ \perp OO^a \wedge YZ \perp XO \Rightarrow YZ \perp (XOO^a)$ and $YZ \perp (XOO^a) \wedge x^a \perp (XOO^a) \Rightarrow x^a \perp YZ$. Similarly $XZ \perp y^a$ and $XY \perp z^a$.

2) Axonometric triangle is acute.

Proof. We show that angle $\xi = \angle XYZ$ is acute, see Fig. 4.1. From the theorem on cosines for triangle ΔXYZ we obtain $|YZ|^2 = |XY|^2 + |XZ|^2 - 2|XY||XZ|\cos\xi$.
From right triangles $\Delta XYO, \Delta XZO, \Delta YZO$ we know that $|YZ|^2 = |XY|^2 + |XZ|^2 - 2|XO|^2$
Comparison of these relation yields and therefore $\cos\xi > 0$. Angle ξ is one of the angles in the triangle ΔXYZ, it is acute.

3) Angles φ, ψ, ω between positive rays of axes x^a, y^a, z^a are obtuse $(90° < \varphi < 180°)$.
See Fig. 4.4

Proof. It follows from (1), (2).

Conclusion
Axonometry is given by axonometric axes or by its axonometric triangle ΔXYZ.

Fig. 4.3

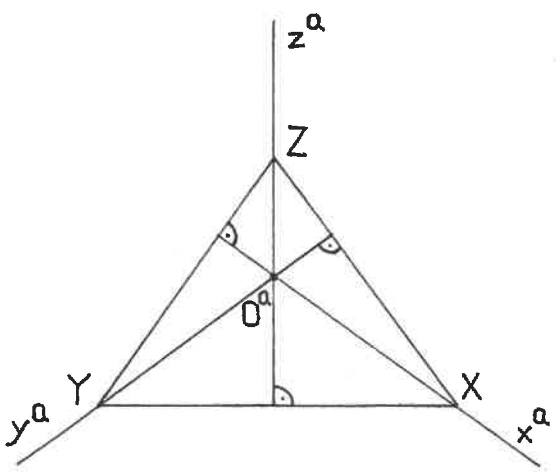
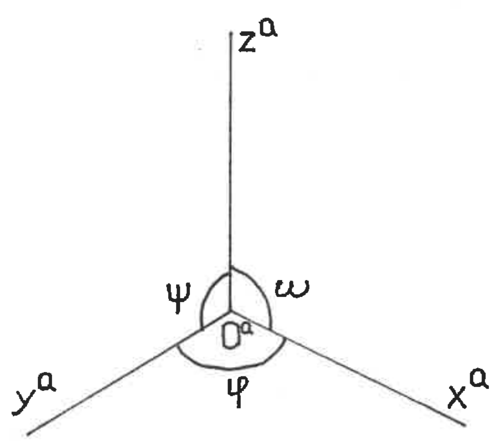


Fig. 4.4



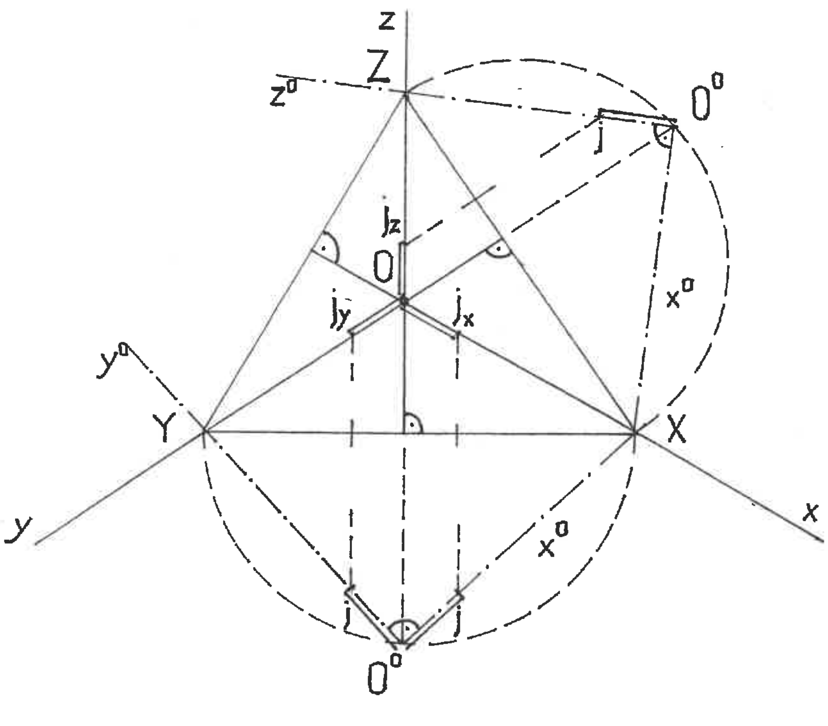
4.2 CONSTRUCTION OF AXONOMETRIC UNITS j_x, j_y, j_z

(views of the unit j on axes).

Axonometry is given by triangle ΔXYZ, unit j is also given.
Solution, see Fig. 4.5. We construct axonometric axes x^a, y^a, z^a, see 4.1, 1) and their intersection O^a (Later let us denote x, y, z, O only). Then we rotate the plane $\pi = (x, y)$ into the picture plane ρ about the axis $o = XY$. Rotated origin O° must lie on Thalet circle (with diameter $d = XY$ and on the line $OZ, OZ \perp XY$. We obtain rotated axes $x^0 = O^0X, y^0 = O^0Y$. The unit j apears on axes x^0, y^0 in true length. Unit j on axes x^0, y^0 is rotated back and we obtain axonometric units j_x, j_y. To obtain j_z we must rotate the plane $\nu = (x, z)$ in the picture plane ρ by means the similar construction with the axis of the rotation $o = XZ$.

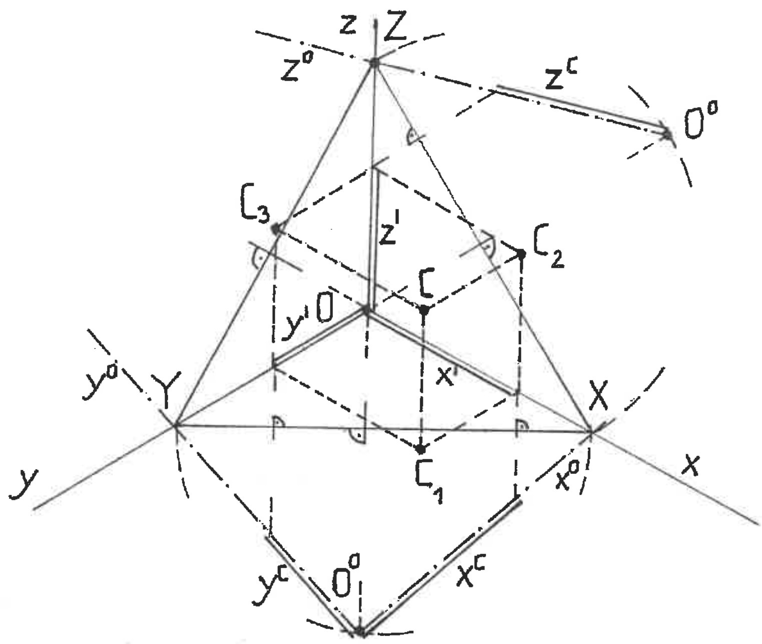
Remark: Using construction 4.2 we can construct axonometric view of coordinates of any point and its coordinate box as well.

Fig. 4.5



4.1 Example: Draw the axonometric projection of the point $C = [x^C, y^C, z^C]$ in the orthogonal axonometry given by the triangle ΔXYZ.
Solution: First we construct rotated axes x^0, y^0, z^0 as shown in Fig. 4.5. and put coordinates x^C, y^C, z^C on them and we rotate them back. We obtain shortened coordinates on axonometric views of coordinate axis x, y, z. We draw axonometric view of the top view C_1 using the parallels with axis x, y (we call this point the axonometric top view). Then the point C lies on the parallel to z-axis and $|CC_1| = z'$ (we assume all segments as oriented line segments). If necessary we complete the coordinate box and find the axonometric front view C_2 and the axonometric side view C_3. See Fig. 4.6.

Fig. 4.6



4.3 TYPES OF ORTHOGONAL AXONOMETRY

We can divide orthogonal axonometries into three groups according to the magnitude of angles φ, ψ, ω between axonometric axes, see Fig. 4.4.

When all three angles are unequal, the axonometry is classified as **a trimetric projection.** When exactly two angles are equal, we speak about **dimetric projection**. For $\varphi = \psi = \omega$ we obtain **isometric projection (isometry).**

4.4 ISOMETRY

4.4.1 Properties of Isometry

1) The axonometric triangle ΔXYZ is equilateral .

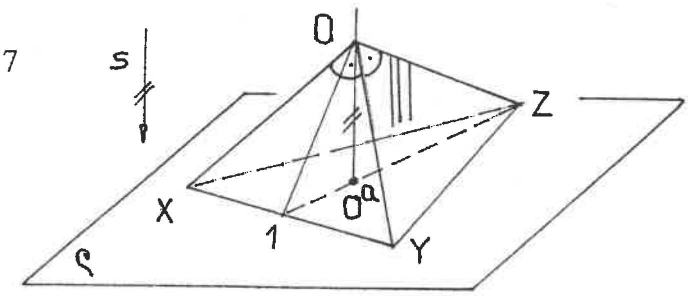
2) For angles between axonometric axes we have $\varphi = \psi = \omega = 120°$.

3) Lengths of segments on axes are shortened by ratio $1 : \sqrt{\frac{2}{3}}$.

Proof. 4.1 (2), (3) yields properties 4.4.1 (1), (2). To prove (3) we observe Fig. 4.7, where $|OX| = |OY| = |OZ| = a$. Using right triangles $\Delta 1OZ$ and ΔXYO we obtain

$$\frac{|O^a Z|}{|OZ|} = \frac{\sqrt{\frac{2}{3}}a}{a} = \sqrt{\frac{2}{3}} \doteq 0.8$$

Fig. 4.7

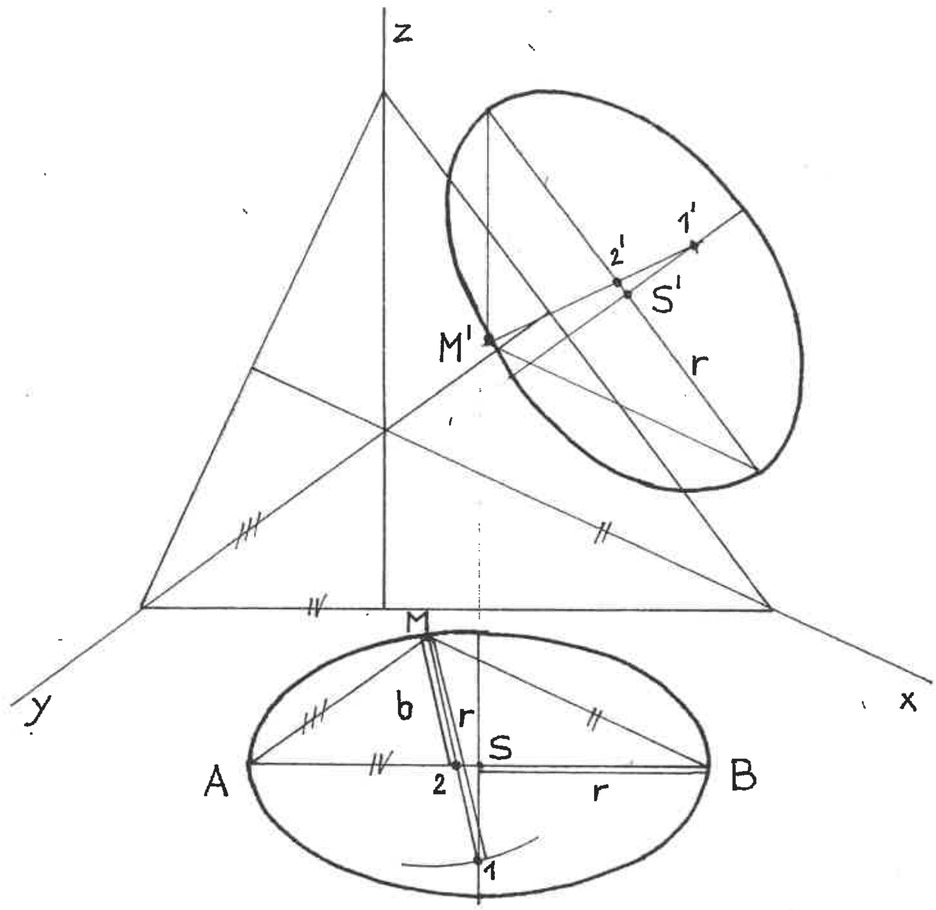


4.5 AXONOMETRY OF A CIRCLE LYING IN THE COORDINATE PLANE

Draw a circle $k \equiv (S, r)$ lying in the coordinate plane $\pi = (x, y)$ in the orthogonal axonometry given by the triangle ΔXYZ. See Fig. 4.8.

The projection of the circle is an ellipse. The diameter AB of the circle is parallel to the axonometric picture plane ρ and so is projected in its true length. The other diameters of the circle are shortened by projecting, so the axonometric view of the segment AB is the major axis of the ellipse. So we construct the segment AB centred at the point S parallel to XY. Its lenght $|AB| = 2r$. Drawing two lines parallel to axes x, y passing through points A and B, we obtain point M of their intersection. Because point M is located on the ellipse (the angle $\angle AMB$ is a right angle, so the point M is located on the Thalet circle with diameter AB and its projection is located on the ellipse), we can use it for the construction of the half minor axis by means of trammel construction, see. Fig. 1.5, Fig. 4.8. The ellipse may be drawn using osculating circles. The construction of the circles in the other coordinate projection planes is similar.

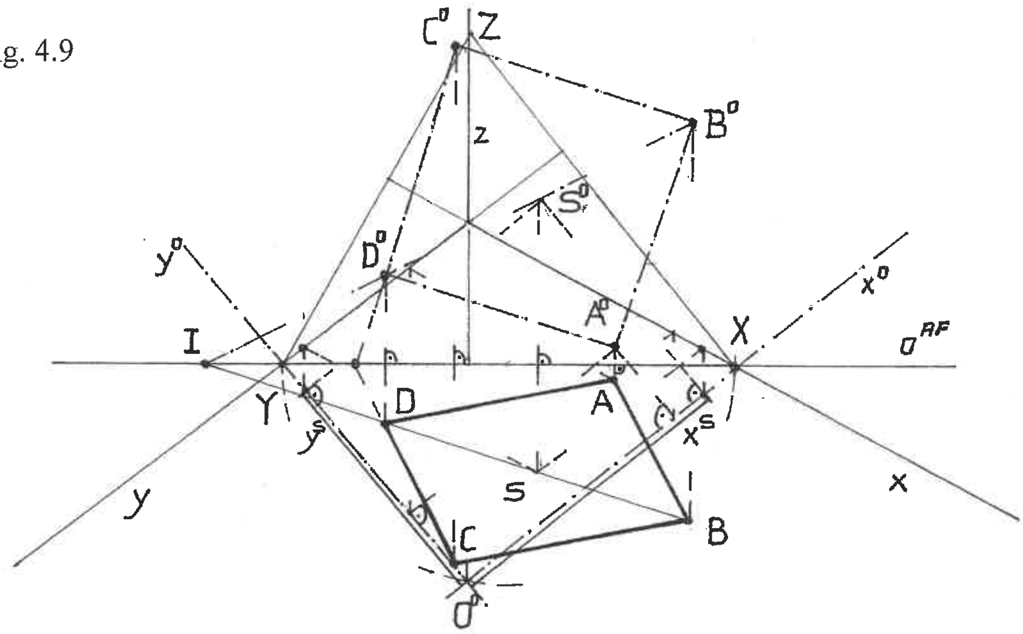
Fig. 4.8



4.6 AXONOMETRY OF A PLANE POLYGON IN THE COORDINATE PLANE

Draw a square $ABCD$ in the plane $\pi = (x, y)$ in the orthogonal axonometry given by the triangle ΔXYZ. Its centre S and point A are given, see Fig. 4.9. We rotate the plane $\pi = (x, y)$ into axonometric picture plane ρ. We obtain rotated axes x^0, y^0 and rotated origin O^0. We construct rotated point S^0 using parallels with x^0, y^0 and we construct S as shown in Fig .4.6. Point A^0 we obtain either similarly like S^0 or using the axial affinity (see 3.9, 3.9.1) with the line XY as the axis of affinity and $S \mapsto S^0$ (or $O \mapsto O^0$) as the pair of corresponding points. Then we draw the rotated square $A^0B^0C^0D^0$. For construction of the axonometry of the square we use the inverse procedure or the axial affinity. The axonometric view of the square is the parallelogram *ABCD*.

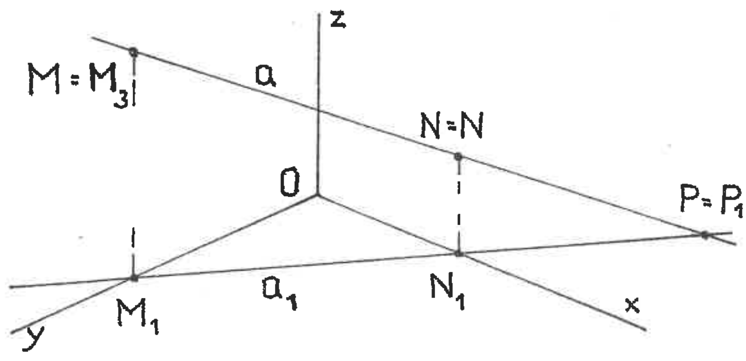
Fig. 4.9



4.7 AXONOMETRY OF AN ARBITRARY STRAIGHT LINE

Let a be an arbitrary nonprojecting straight line. The intersection of a with the coordinate planes are called the trace points (the intersection with the picture plane ρ is called axonometric trace point but it is not necessary for constructions here). See Fig. 4.10. $P = a \cap \pi, \pi = (x, y)$, $N = a \cap \nu, \nu = (x, z)$, $M = a \cap \mu, \mu = (y, z)$

Fig. 4.10



4.8 SPECIAL POSITIONS OF STRAIGHT LINES

Fig. 4.11 $a \perp \pi, b \perp \nu, c \perp \mu$

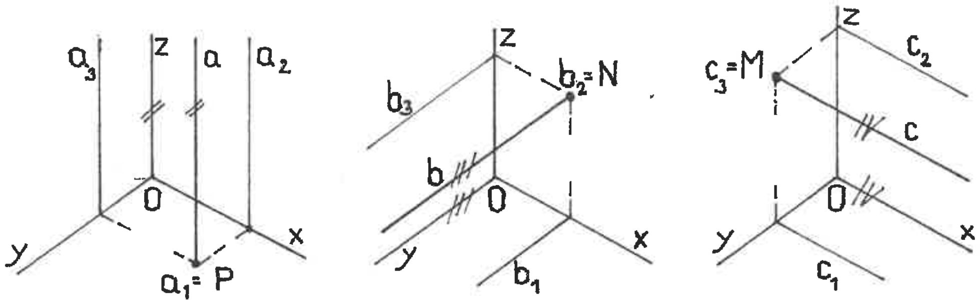
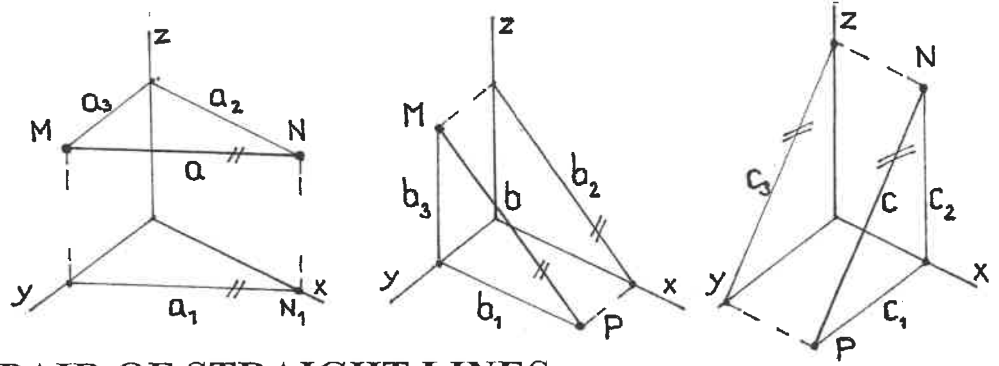


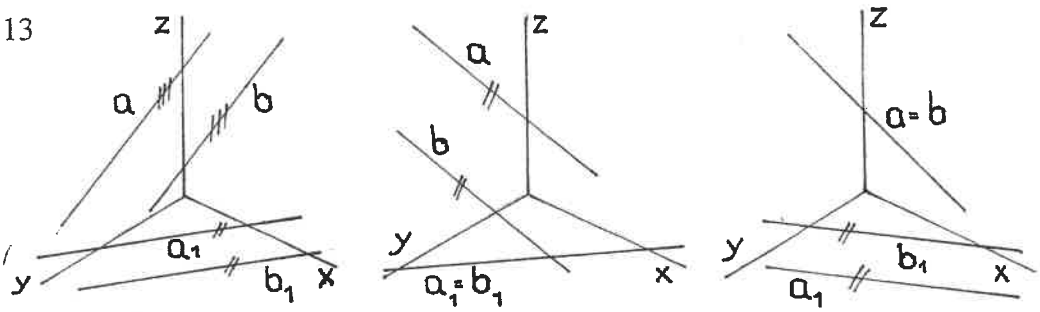
Fig. 4.12 $a \| \pi$, b$\| \nu$, $c \| \mu$



4.9 A PAIR OF STRAIGHT LINES

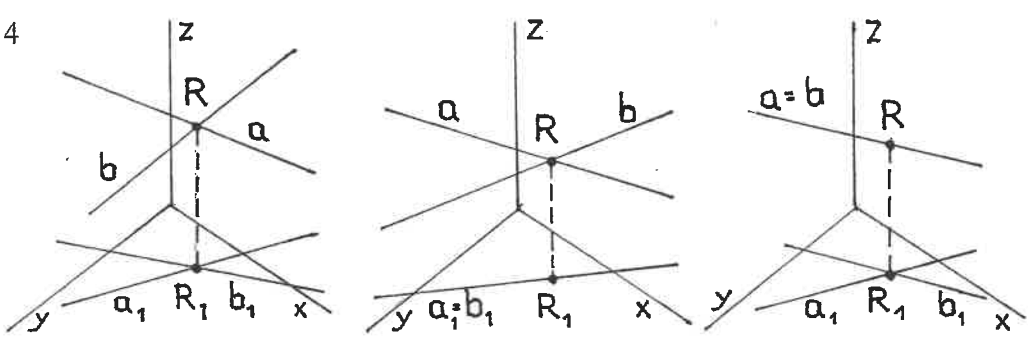
4.9.1 Parallels

Fig. 4.13



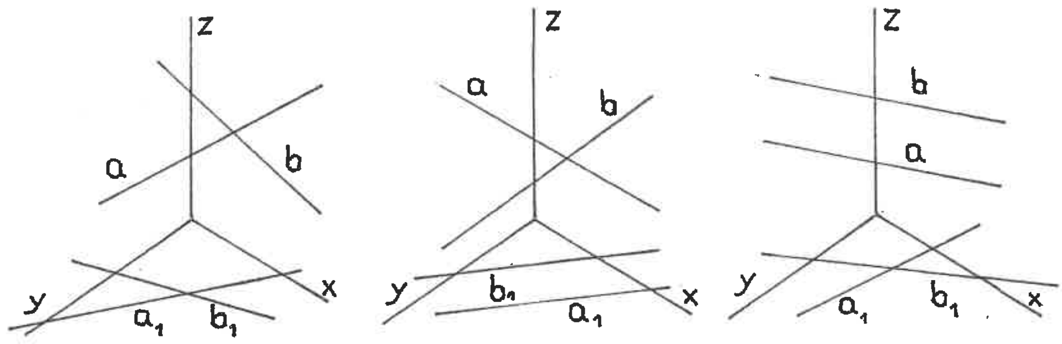
4.9.2 Intersecting Lines

Fig. 4.14



4.9.3 Skew Lines

Fig. 4.15



4.10 SPECIAL POSITIONS OF A PLANE

Special positions of a plane defined by means of **traces**, intersection lines of a given plane and the coordinate planes, denoted by $p^{\alpha}, n^{\alpha}, m^{\alpha}$ for a given plane α . See Fig. 4.16, Fig. 4.17.

Fig. 4.16 $\alpha \perp \pi, \beta \perp \nu, \gamma \perp \mu$

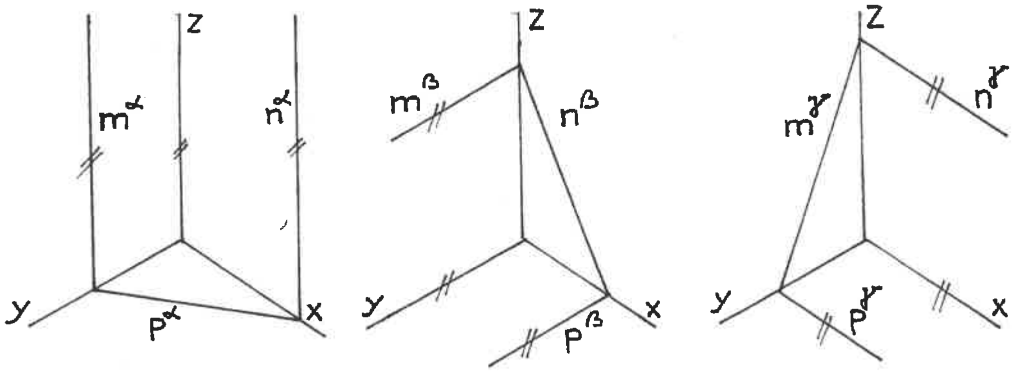
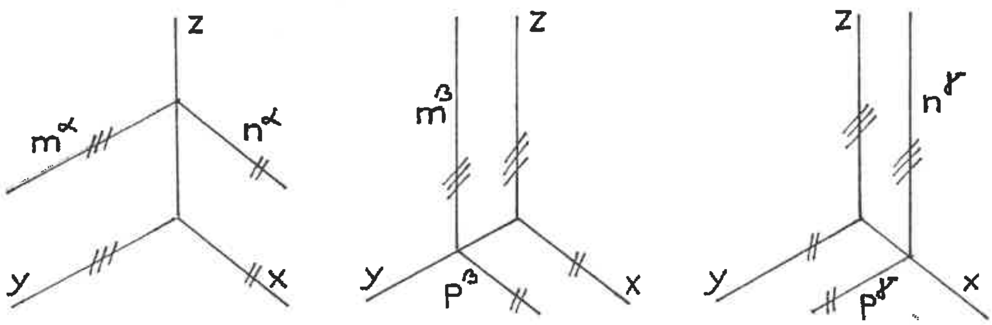


Fig. 4.17 $\alpha \parallel \pi, \beta \parallel \nu, \gamma \parallel \mu$



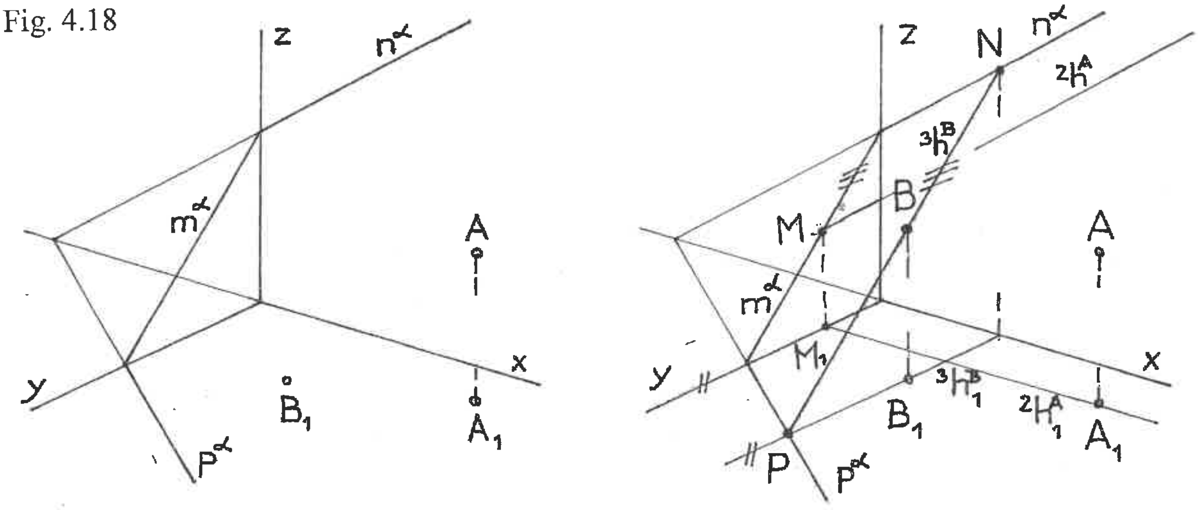
4.11 A POINT IN A PLANE

Locate a point B, its axonometric top view is given, on the plane α. Verify the point A is lying in the given plane α. See Fig. 4.18.

Solution: If a straight line lies in the plane then its trace points lie on the trace lines of this plane. Locate the point B first. Assume a line ${}^3h^B$ passing through the point B and lying in the plane α. The axonometric top view ${}^3h_1^B$ passes through the point B_1 and it is parallel to the y-axis. Find the trace points P and N on the trace lines p^α, n^α and draw the line ${}^3h^B$. We locate the point B on this line, $BB_1 \parallel z$.

The same procedure is used in the second case. The line ${}^2h_1^A$ passes through the point A_1 and it is parallel to the x-axis. We suppose the line ${}^2h^A$ lies on the plane α and we draw the axonometric view ${}^2h^A$, ${}^2h^A \parallel n^\alpha$. The point A does not lie on ${}^2h^A$ and so A does not lie either in the plane α.

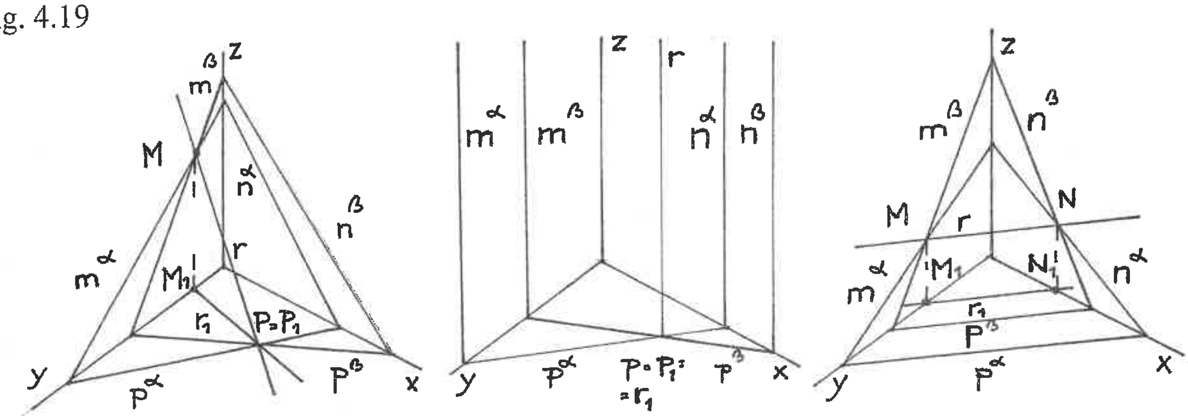
Fig. 4.18



4.12 INTERSECTION OF TWO PLANES

The intersection line of two planes is defined by the common points of trace lines as shown in the Fig. 4.19.

Fig. 4.19

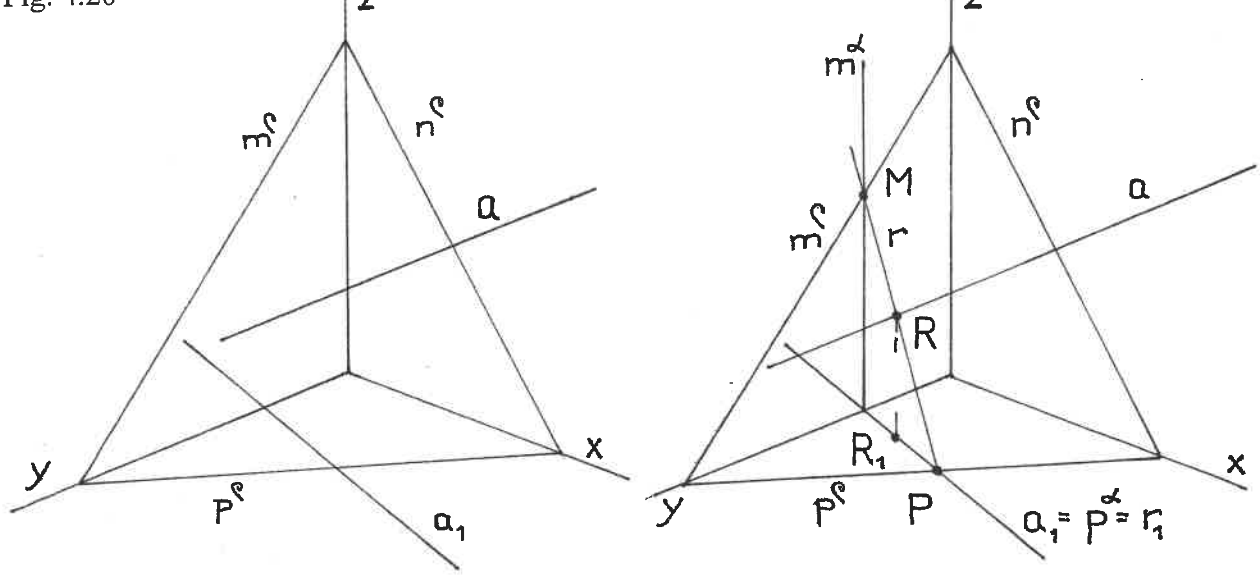


4.13 AN INTERSECTION OF A STRAIGHT LINE AND A PLANE

Find the intersection point of the line a and the plane ρ.

Solution: The method of finding the common point of a line and a plane is described in 3.14. Assume the plane $\alpha, a \in \alpha$. The construction is easy when α is perpendicular to $\pi = (x, y)$. Then find the intersection line $r = \alpha \cap \rho$ as shown in Fig. 4.19. Construct the intersection point R of r and a.(If $a \| r$ then $a \| \rho$). See Fig.4.20.

Fig. 4.20



REFERENCES

Kargerová, M..: Geometry and Computer Graphics, ČVUT Praha, 1998
Černý, J.: Geometry, ČVUT Praha, 1996
Urban, A.: Deskriptívní geometrie I, II, SNTL Praha, 1967

Author:	Eva Vavříková Faculty of Mining and Geology
Title:	Descriptive Geometry
Place, Year, Edition:	Ostrava, 2005, 1st edition
Page number:	39
Issued by:	VŠB – Technical University of Ostrava 17. listopadu 15/2172 708 33 Ostrava - Poruba
Printed by:	Editing Centre of VŠB – Technical University of Ostrava, CR
Impression:	30

For sale

ISBN 80 – 248 – 1006 - 9