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# Worksheets for Mathematics III

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## **Introduction**

The study material is designed for students of VSB - Technical University of Ostrava.

The worksheets consist of several theoretical sheets, some solved problems and some sheets with unsolved problems for practicing. The materials should support classwork and they are not recommended for self-study or as a replacement for textbooks.

## **Thanks**

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# Contents

	Statical moments and moments of inertia . . . . .	97	
<b>Double integral</b>	<b>4</b>	<b>Theory of the field</b>	<b>102</b>
Double integral over rectangular domain . . . . .	5	Vector function . . . . .	103
Double integral over a general domain . . . . .	14	Scalar field . . . . .	106
Double integral in polar coordinates . . . . .	24	Vector field . . . . .	112
Double integral in generalized polar coordinates . . . . .	33	<b>Line integral</b>	<b>120</b>
Practical applications of the double integral . . . . .	36	The curve and its parametrization . . . . .	121
Area of a region . . . . .	36	Parametrization of the line segment . . . . .	124
Volume of a body . . . . .	41	Parametrization of the circle . . . . .	125
Surface area . . . . .	47	Parametrization of the ellipse . . . . .	126
Center of mass . . . . .	52	Line integral of a scalar field . . . . .	127
<b>Triple integral</b>	<b>55</b>	Practical applications of line integral of the scalar field . . .	134
Triple integral over rectangular hexahedron . . . . .	56	Area of a cylindrical region . . . . .	134
Triple integral over a general domain . . . . .	63	Length of a curve . . . . .	137
Transformation of the triple integral . . . . .	72	Mass of a curve . . . . .	140
Transformation to cylindrical coordinates . . . . .	73	Line integral of a vector field . . . . .	143
Transformation to spherical coordinates . . . . .	80	Green's theorem . . . . .	149
Practical applications of the triple integral . . . . .	86	Path independence of line integral . . . . .	156
Volume of a body . . . . .	86	Practical applications of line integral of the vector field . .	164
Mass of a body . . . . .	94	Work in a force field . . . . .	164

# **Worksheets for Mathematics III**

Double integral

## 5 – Double integral over rectangular domain

As the definite integral of a continuous positive function of one variable represents the area of the region between the graph and the  $x$ -axis, the **double integral** of a continuous positive function of two variables represents the volume of the region between the surface defined by the function  $z = f(x, y)$  and the  $xy$ -plane which contains its domain. We start with rectangular domain

$$D = \{[x, y] \in \mathbb{R}^2 : x \in [a, b], y \in [c, d]\}$$

on the  $xy$ -plane according to figure.

We divide interval  $[a, b]$ , resp.  $[c, d]$  by sequences of points

$$a = x_0 < x_1 < x_2 < \dots < x_m = b,$$

resp.

$$c = y_0 < y_1 < y_2 < \dots < y_n = d$$

to intervals  $[x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, m$ , resp.  $[y_{j-1}, y_j]$ ,  $j = 1, 2, \dots, n$ . We denote sizes of each component  $\Delta x_i = x_i - x_{i-1}$ ,  $\Delta y_j = y_j - y_{j-1}$ .

This way is the whole rectangular domain divided into  $m \cdot n$  small rectangles with area  $\Delta D_{ij} = \Delta x_i \cdot \Delta y_j$ . Now we can choose an arbitrary point  $[\xi_i, \eta_j]$  in each rectangle  $D_{ij}$  and we can evaluate the volume of a prism with basis  $D_{ij}$  and height  $z = f(\xi_i, \eta_j)$ . The sum of the volumes

$$\sum_{i=1}^m \sum_{j=1}^n f(\xi_i, \eta_j) \cdot \Delta x_i \cdot \Delta y_j$$

represents the volume of the body consisted of such prisms over all rectangles  $D_{ij}$  if  $f(x, y) \geq 0$  on  $D$ .

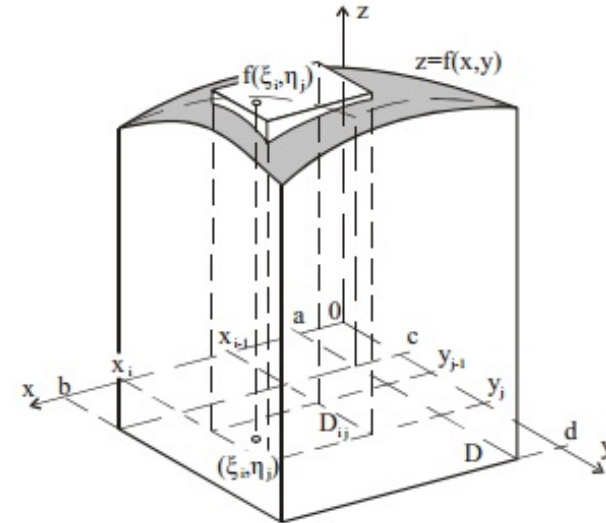
### Definition

If there exists

$$\lim \sum_{i=1}^m \sum_{j=1}^n f(\xi_i, \eta_j) \Delta x_i \Delta y_j$$

for  $m \rightarrow \infty, n \rightarrow \infty, \Delta x_i \rightarrow 0, \Delta y_j \rightarrow 0$  for all  $i = 1, 2, \dots, m, j = 1, 2, \dots, n$ , we call it **the double integral** of a function  $f(x, y)$  over the rectangular domain  $D$  and denote it

$$\iint_D f(x, y) dx dy.$$



## 6 – Double integral over rectangular domain

**Theorem (Fubini's theorem)**

Let  $D = \{[x, y] \in \mathbb{R}^2: x \in [a, b], y \in [c, d]\}$ . If function  $f(x, y)$  is continuous on rectangle  $D$ , then

$$\iint_D f(x, y) \, dx \, dy = \int_a^b \left( \int_c^d f(x, y) \, dy \right) dx = \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy.$$

In fact there are two ways of computing the double integral. If the inner differential is  $dy$  then the limits of the inner integral must have  $y$  limits of integration and outer integral must have  $x$  limits of integration. We calculate the integral  $\int_c^d f(x, y) \, dy$  by holding  $x$  constant and integrating with respect to  $y$  as if this were a single integral (similar approach is used for partial derivatives of function of more than one variable). This will result as a function of a single variable  $x$  which can be integrated once again. We use similar approach for the second way of computing the double integral.

We usually write

$$\int_a^b \left( \int_c^d f(x, y) \, dy \right) dx = \int_a^b dx \int_c^d f(x, y) \, dy$$

and

$$\int_c^d \left( \int_a^b f(x, y) \, dx \right) dy = \int_c^d dy \int_a^b f(x, y) \, dx.$$

## 7 – Double integral over rectangular domain

**Theorem (Properties of the double integral over a rectangular domain)**

$$1. \iint_D cf(x, y) \, dx \, dy = c \iint_D f(x, y) \, dx \, dy,$$

$$2. \iint_D (f(x, y) + g(x, y)) \, dx \, dy = \iint_D f(x, y) \, dx \, dy + \iint_D g(x, y) \, dx \, dy,$$

$$3. \iint_D f(x, y) \, dx \, dy = \iint_{D_1} f(x, y) \, dx \, dy + \iint_{D_2} f(x, y) \, dx \, dy,$$

where  $f, g$  are continuous functions on  $D$ ,  $c \in \mathbb{R}$  and  $D_1, D_2$  are non-overlapping rectangles that fulfil  $D = D_1 \cup D_2$ .

## 8 – Double integral over rectangular domain

**Example**

Compute  $I = \iint_D (2xy + 4x) \, dx \, dy$  over the domain  $D: 0 \leq x \leq 2, -1 \leq y \leq 3$ .

We will show both ways of the computing

a) by integrating the inner integral with respect to variable  $x$

$$\begin{aligned} I &= \iint_D (2xy + 4x) \, dx \, dy = \int_{-1}^3 \left( \int_0^2 (2xy + 4x) \, dx \right) dy \\ &= \int_{-1}^3 [x^2y + 2x^2]_0^2 dy = \int_{-1}^3 (4y + 8) \, dy = [2y^2 + 8y]_{-1}^3 = 48 \end{aligned}$$

b) by integrating the inner integral with respect to variable  $y$

$$\begin{aligned} I &= \int_0^2 \left( \int_{-1}^3 (2xy + 4x) \, dy \right) dx = \int_0^2 [xy^2 + 4xy]_{-1}^3 dx \\ &= \int_0^2 ((9x + 12x) - (x - 4x)) \, dx = \int_0^2 24x \, dx = [12x^2]_0^2 = 48 \end{aligned}$$

If the integrand  $f(x, y)$  can be written as a multiplication of two functions of one variable  $f(x, y) = f_1(x) \cdot f_2(y)$ , then it holds:

$$\iint_D f(x, y) \, dx \, dy = \int_a^b f_1(x) \, dx \cdot \int_c^d f_2(y) \, dy.$$

Compute the integral by using decomposition on two functions of one variable.

$$\begin{aligned} I &= \iint_D 2x(y + 2) \, dx \, dy = \int_0^2 2x \, dx \cdot \int_{-1}^3 (y + 2) \, dy \\ &= [x^2]_0^2 \cdot \left[ \frac{y^2}{2} + 2y \right]_{-1}^3 = 4 \cdot \left[ \left( \frac{9}{2} + 6 \right) - \left( \frac{1}{2} - 2 \right) \right] = 48 \end{aligned}$$

**Remark**

If the decomposition is not possible, we can always use Fubini's theorem.



## 9 – Double integral over rectangular domain

## Example

Compute  $I = \iint_D x \sqrt{x^2 + y} \, dx \, dy$  over the domain  $D: 0 \leq x \leq 1, 0 \leq y \leq 3$ .

$$\begin{aligned} I &= \int_0^3 dy \int_0^1 x \sqrt{x^2 + y} \, dx = \left| \begin{array}{l} t = x^2 + y \quad 0 \rightarrow y \\ dt = 2x \, dx \quad 1 \rightarrow y + 1 \end{array} \right| = \frac{1}{2} \int_0^3 dy \int_y^{y+1} \sqrt{t} \, dt = \frac{1}{2} \int_0^3 \left[ \frac{2}{3} \sqrt{t^3} \right]_y^{y+1} dy \\ &= \frac{1}{3} \int_0^3 \left( \sqrt{(y+1)^3} - \sqrt{y^3} \right) dy = \frac{1}{3} \left[ \frac{2}{5} \sqrt{(y+1)^5} - \frac{2}{5} \sqrt{y^5} \right]_0^3 = \frac{2}{15} (32 - 9\sqrt{3} - 1) = \frac{2}{15} (31 - 9\sqrt{3}). \end{aligned}$$

## 10 – Double integral over rectangular domain

**Example**

Compute  $I = \iint_D (2x^2y + y^3) \cos x \, dx \, dy$  over the domain  $D: 0 \leq x \leq \frac{\pi}{2}, -1 \leq y \leq 1$ .

**Remark**

Although generally the order of integration doesn't matter, in some cases the integral can be easily solved by using one way of integration while it can be rather complicated using the other way. Everything depends on the integrand  $f(x, y)$  itself and on the limits of integration.

a) First we integrate the inner integral with respect to variable  $x$

$$\begin{aligned} I &= \int_{-1}^1 dy \int_0^{\frac{\pi}{2}} (2x^2y + y^3) \cos x \, dx = \left| \begin{array}{ll} u = 2x^2y + y^3 & v' = \cos x \\ u' = 4xy & v = \sin x \end{array} \right| \\ &= \int_{-1}^1 \left( \left[ (2x^2y + y^3) \sin x \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 4xy \sin x \, dx \right) dy = \left| \begin{array}{ll} u = 4xy & v' = \sin x \\ u' = 4y & v = -\cos x \end{array} \right| \\ &= \int_{-1}^1 \left( \frac{\pi^2}{2}y + y^3 - [-4xy \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} -4y \cos x \, dx \right) dy = \int_{-1}^1 \left( \frac{\pi^2}{2}y + y^3 - 4y [\sin x]_0^{\frac{\pi}{2}} \right) dy \\ &= \int_{-1}^1 \left( \frac{\pi^2}{2}y + y^3 - 4y \right) dy = \left[ \frac{\pi^2}{4}y^2 + \frac{y^4}{4} - 2y^2 \right]_{-1}^1 = \frac{\pi^2}{4} + \frac{1}{4} - 2 - \left( \frac{\pi^2}{4} + \frac{1}{4} - 2 \right) = 0. \end{aligned}$$

b) Now we integrate the inner integral with respect to variable  $y$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} dx \int_{-1}^1 (2x^2y + y^3) \cos x \, dy = \int_0^{\frac{\pi}{2}} \left[ \left( x^2y^2 + \frac{y^4}{4} \right) \cos x \right]_{-1}^1 dx \\ &= \int_0^{\frac{\pi}{2}} \left( \left( x^2 + \frac{1}{4} \right) \cos x - \left( x^2 + \frac{1}{4} \right) \cos x \right) dx = \int_0^{\frac{\pi}{2}} 0 \, dx = 0. \end{aligned}$$

**Hints****By Parts**

$$\begin{aligned} &\int_a^b u(x) \cdot v'(x) \, dx \\ &= [u(x) \cdot v(x)]_a^b - \int_a^b u'(x) \cdot v(x) \, dx \end{aligned}$$

## 11 – Double integral over rectangular domain

## Exercise

Compute following integrals over their domains  $D$ .

a)  $\iint_D \sqrt{5x+4} \ln y \, dx \, dy, \quad D: 0 \leq x \leq 1, 1 \leq y \leq 3$

b)  $\iint_D (x^2 + y^2) \, dx \, dy, \quad D: -2 \leq x \leq 0, -1 \leq y \leq 2$

## 12 – Double integral over rectangular domain

### Exercise

Compute integral  $\iint_D \sin(2x + y) \, dx \, dy$  over domain  $D: 0 \leq x \leq \pi, \frac{\pi}{4} \leq y \leq \pi$ .

## 13 – Double integral over rectangular domain

### Exercise

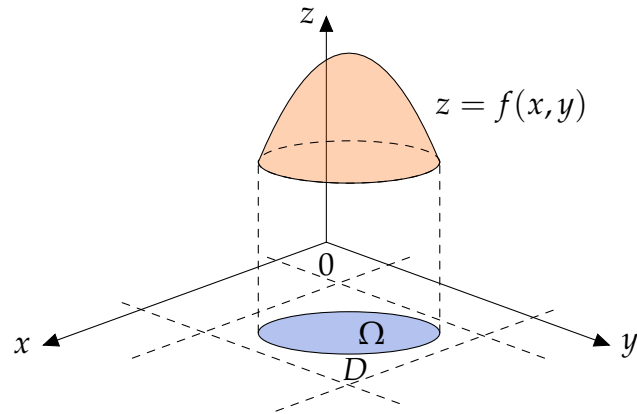
Compute integral  $\iint_D \frac{1}{(x+y+1)^2} dx dy$  over domain  $D: 0 \leq x \leq 1, 0 \leq y \leq 1$ .

## 14 – Double integral over a general domain

There is no reason to limit our problem to rectangular regions. The integral domain can be of a general shape. We extend the Riemann's definition of the double integral over rectangular domain to a closed connected bounded domain  $\Omega$  without any problem. The domain is connected if we can connect every two points from it by curve that lies within the domain. We can always find a rectangle  $D$  that fulfils  $\Omega \subseteq D$  and we can define function  $f^*(x, y)$  by

$$f^*(x, y) = \begin{cases} f(x, y) & \forall [x, y] \in \Omega, \\ 0 & \forall [x, y] \in D \setminus \Omega. \end{cases}$$

Then it holds  $\iint_{\Omega} f(x, y) \, dx \, dy = \iint_D f^*(x, y) \, dx \, dy$ .



The properties of the double integral over a general domain must correspond to next Theorem:

**Theorem (Properties of the double integral over a general domain)**

1.  $\iint_{\Omega} c f(x, y) \, dx \, dy = c \iint_{\Omega} f(x, y) \, dx \, dy,$
2.  $\iint_{\Omega} (f(x, y) + g(x, y)) \, dx \, dy$   
 $= \iint_{\Omega} f(x, y) \, dx \, dy + \iint_{\Omega} g(x, y) \, dx \, dy,$
3.  $\iint_{\Omega} f(x, y) \, dx \, dy = \iint_{\Omega_1} f(x, y) \, dx \, dy + \iint_{\Omega_2} f(x, y) \, dx \, dy,$

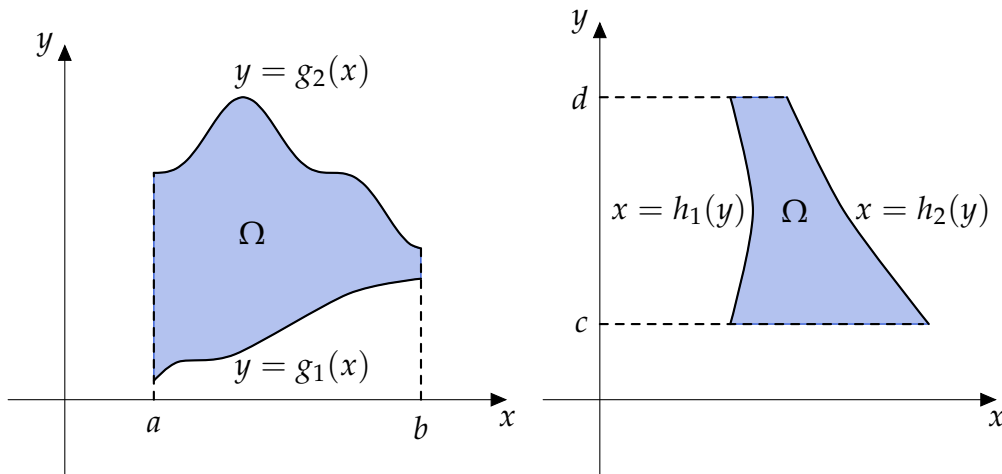
where  $f, g$  are continuous functions on  $\Omega$ ,  $c \in \mathbb{R}$  and  $\Omega_1, \Omega_2$  are non-overlapping domains that fulfil  $\Omega = \Omega_1 \cup \Omega_2$ .

## 15 – Double integral over a general domain

There are two types of domains we need to look at.

**Definition**

1. **Normal domain with respect to the  $x$ -axis** is bounded by lines  $x = a$ ,  $x = b$ , where  $a < b$ , and continuous curves  $y = g_1(x)$ ,  $y = g_2(x)$ , where  $g_1(x) < g_2(x)$ , for all  $x \in [a, b]$ .
2. **Normal domain with respect to the  $y$ -axis** is bounded by lines  $y = c$ ,  $y = d$ , where  $c < d$ , and continuous curves  $x = h_1(y)$ ,  $x = h_2(y)$ , where  $h_1(y) < h_2(y)$ , for all  $y \in [c, d]$ .

**Theorem (Fubini's theorem)**

1. If the function  $f(x, y)$  is continuous on a domain that is normal with respect to the  $x$ -axis, then it holds

$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy.$$

2. If the function  $f(x, y)$  is continuous on a domain that is normal with respect to the  $y$ -axis, then it holds

$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_c^d dy \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx.$$

## 16 – Double integral over a general domain

**Example**

Determine integration limits for  $\iint_{\Omega} f(x, y) \, dx \, dy$  over the domain  $\Omega$ , which is bounded by curves  $y^2 = 2x$  and  $x = 2$ .

We need to find intersections of curves  $y^2 = 2x$  and  $x = 2$  by solving the system of these two equations. We can eliminate variable  $x$ , receive equation  $y^2 = 4$  and solve it. We obtain two solutions  $y_1 = 2$ ,  $y_2 = -2$ . Given curves intersects each other in points  $[2, -2]$  and  $[2, 2]$ . Treating the domain  $\Omega$  as a normal with respect to the  $x$ -axis, we can see the domain is bounded by  $0 \leq x \leq 2$ , while limits for variable  $y$  must be obtained from the equation  $y^2 = 2x$ . Therefore  $y = \pm\sqrt{2x}$ .

We can express inequalities for  $\Omega$  in the form:

$$\Omega: \quad 0 \leq x \leq 2, \\ -\sqrt{2x} \leq y \leq \sqrt{2x}$$

and according to Fubini's theorem

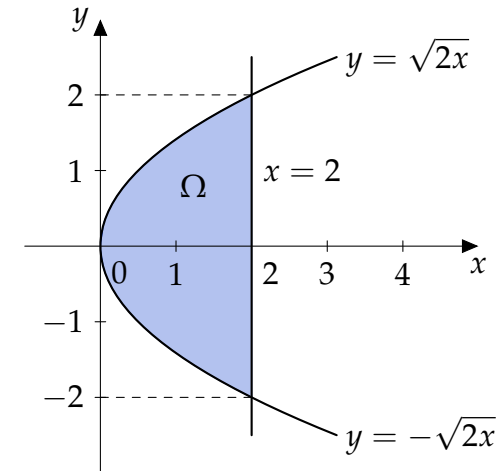
$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} f(x, y) \, dy.$$

We can use a similar procedure and express the integral as an integral over normal domain with respect to the  $y$ -axis with inequalities

$$\Omega: \quad -2 \leq y \leq 2, \\ \frac{y^2}{2} \leq x \leq 2.$$

The double integral then takes form

$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_{-2}^2 dy \int_{\frac{y^2}{2}}^2 f(x, y) \, dx.$$





## 17 – Double integral over a general domain

## Example

Determine integration limits for  $\iint_{\Omega} f(x, y) \, dx \, dy$  over the domain  $\Omega$ , which is a triangle  $ABC$ , where  $A = [-3, 1]$ ,  $B = [5, 1]$ ,  $C = [1, 5]$ .

## Remark

Lines can be described algebraically by linear equations  $y = ax + b$ . We substitute coordinates of points  $A$  and  $C$  to the equation and we obtain system of two linear equations from which we calculate  $a$  and  $b$ :

$$A : 1 = -3a + b$$

$$C : 5 = a + b$$

We get  $a = 1$ ,  $b = 4$  and  $y = x + 4$ .

First, we express the domain as normal with respect to the  $x$ -axis. If we bound the domain by  $-3 \leq x \leq 5$ , the upper limit of inner integral can't be written as one curve and we need to divide the domain  $\Omega$  into two subdomains  $\Omega_1, \Omega_2$  by line  $x = 1$ :

$$\Omega_1: \begin{aligned} -3 &\leq x \leq 1, \\ 1 &\leq y \leq x + 4, \end{aligned}$$

$$\Omega_2: \begin{aligned} 1 &\leq x \leq 5, \\ 1 &\leq y \leq 6 - x. \end{aligned}$$

Using Fubini's theorem we can express

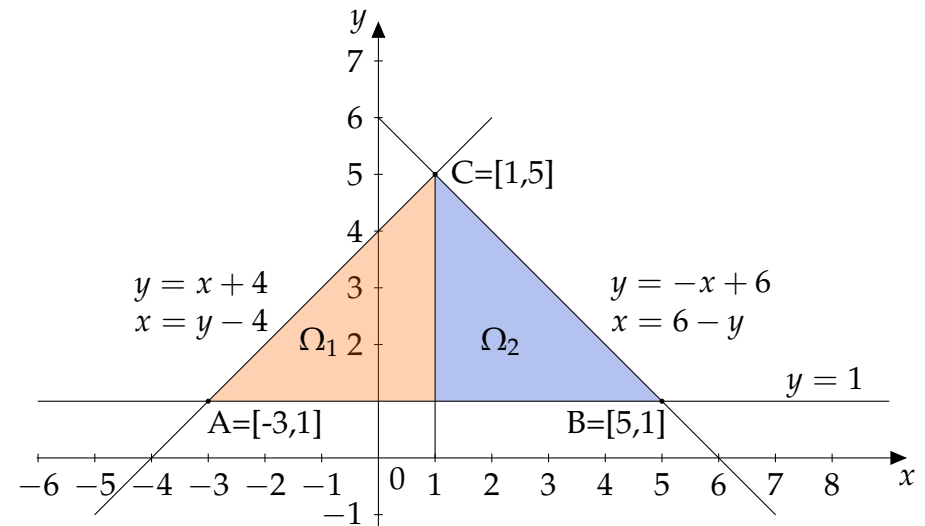
$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_{-3}^1 dx \int_1^{x+4} f(x, y) \, dy + \int_1^5 dx \int_1^{6-x} f(x, y) \, dy.$$

However, it is much better to express the domain as normal with respect to the  $y$ -axis. There is no reason to split the domain which is now bounded by inequalities

$$\Omega: \begin{aligned} 1 &\leq y \leq 5, \\ y - 4 &\leq x \leq 6 - y, \end{aligned}$$

where we have expressed a variable  $x$  from boundary equations. The integral is written in the form

$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_1^5 dy \int_{y-4}^{6-y} f(x, y) \, dx.$$

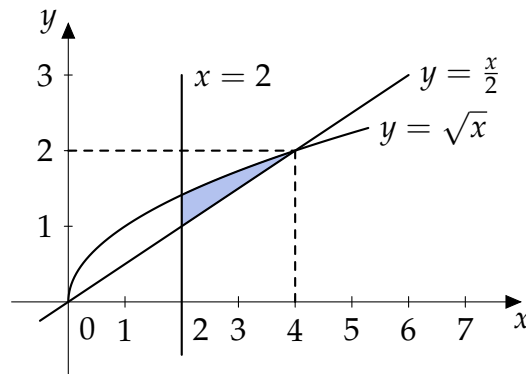


## 18 – Double integral over a general domain

## Example

Compute  $\iint_{\Omega} xy \, dx \, dy$ ,  $\Omega$  is bounded by  $y = \frac{x}{2}$ ,  $y = \sqrt{x}$ ,  $x \geq 2$ .

Solving the system of equations  $y = \frac{x}{2}$ ,  $y = \sqrt{x}$  we receive intersections of both curves in  $x = 0$ ,  $x = 4$ .



It is better to express the domain as normal with respect to  $x$ -axis with boundaries

$$\Omega: \quad 2 \leq x \leq 4, \\ \frac{x}{2} \leq y \leq \sqrt{x}$$

and compute the integral

$$\iint_{\Omega} xy \, dx \, dy = \int_2^4 dx \int_{x/2}^{\sqrt{x}} xy \, dy = \int_2^4 x \left[ \frac{y^2}{2} \right]_{x/2}^{\sqrt{x}} dx = \int_2^4 \left( \frac{x^2}{2} - \frac{x^3}{8} \right) dx = \left[ \frac{x^3}{6} - \frac{x^4}{32} \right]_2^4 = \frac{11}{6}.$$

The second approach requires splitting the domain into two subdomains. It is a good exercise to compute the example this way.

## 19 – Double integral over a general domain

### Exercise

Compute integral  $\iint_{\Omega} (5x^2 - 2xy) \, dx \, dy$ . Domain  $\Omega$  is triangle  $ABC$ , where  $A = [0, 0]$ ,  $B = [2, 0]$ ,  $C = [0, 1]$ .

## 20 – Double integral over a general domain

### Exercise

Compute integral  $\iint_{\Omega} x^2 \, dx \, dy$  over domain  $\Omega: y = \frac{16}{x}, y = x, x = 8$ .

## 21 – Double integral over a general domain

### Exercise

Compute integral  $\iint_{\Omega} 6xy \, dx \, dy$  over domain  $\Omega: y = 0, x = 2, y = x^2$ .

## 22 – Double integral over a general domain

### Exercise

Compute integral  $\iint_{\Omega} xy \, dx \, dy$  over domain  $\Omega: x^2 + 4y^2 \leq 4, x \geq 0, y \geq 0$ .

## 23 – Double integral over a general domain

### Exercise

Compute integral  $\iint_{\Omega} (1 - 2x - 3y) \, dx \, dy$  over domain  $\Omega: x^2 + y^2 \leq 2$ .

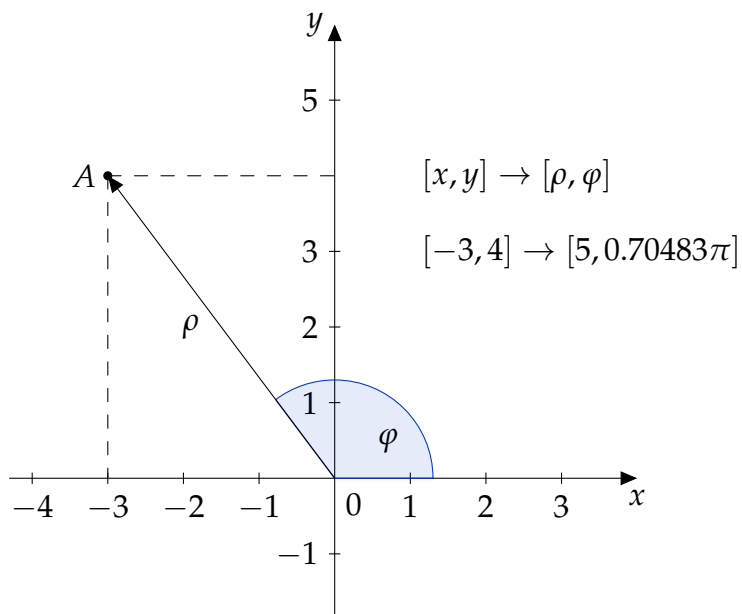
## 24 – Double integral in polar coordinates

At this moment we are able to compute the double integral over a general domain. In this section we want to look at some domains that are easier to describe in a terms of polar coordinates. We might have a domain that is a disc, ring or part of a disc or ring. Let us consider a double integral of an arbitrary function over the disc with the center in the origin of coordinates and with the radius  $r = 2$  (same domain that is used in last Exercise). Using Cartesian coordinates we obtain limits of the integral

$$\Omega: \quad \begin{aligned} -2 \leq x \leq 2, \\ -\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}. \end{aligned}$$

and by Fubini's theorem the integral can be written in the form

$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_{-2}^2 dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x, y) \, dy.$$



In such cases using Cartesian coordinates can be tedious. However, we are able to replace Cartesian coordinates  $x, y$  by polar coordinates  $\rho, \varphi$ , where  $\rho$  denotes a distance between the point  $[x, y]$  and the origin of coordinates and is called a **radius**, and,  $\varphi$  denotes the positively oriented angle between positive part of the  $x$ -axis and the radius vector and is called **angular coordinate** or **azimuth**.

The transformation to cylindrical coordinates is given by transformation equations

$$\begin{aligned} x &= \rho \cos \varphi, \\ y &= \rho \sin \varphi. \end{aligned}$$

Transformation to polar coordinates is a special case of mapping region  $\Omega$  onto  $\Omega^*$  that is an image of  $\Omega$  in polar coordinates in our case. For example a disc with the center in the origin of coordinates and with the radius  $r = 2$ ,

$$\Omega = \{[x, y]: x^2 + y^2 \leq 4\},$$

is mapped onto

$$\Omega^* = \{[\rho, \varphi]: \rho \in (0, 2], \varphi \in [0, 2\pi)\}.$$



## 25 – Double integral in polar coordinates

**Theorem (Transformation to general coordinates)**

- Let equations  $x = u(r, s)$ ,  $y = v(r, s)$  map the region  $\Omega$  bijectively to the region  $\Omega^*$ .
- Let function  $f(x, y)$  be continuous and bounded on  $\Omega$  and functions  $x = u(r, s)$ ,  $y = v(r, s)$  have continuous partial derivatives on  $\hat{\Omega}$  that fulfils  $\Omega^* \subset \hat{\Omega}$ .

- Let  $J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \neq 0$  in  $\Omega^*$ .

Then

$$\iint_{\Omega} f(x, y) \, dx \, dy = \iint_{\Omega^*} f(u(r, s), v(r, s)) |J(u, v)| \, dr \, ds.$$

Determinant

$$J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix}$$

is called **Jacobian** or **Jacobi determinant**.

We will use this theorem for transformation of the double integral to polar coordinates as well as the triple integral to cylindrical and spherical coordinates.

According to the theorem we replace square element  $dx \, dy$  by  $|J| \, d\rho \, d\varphi$ , where the Jacobian of the transformation to polar coordinates satisfies

$$J(\rho, \varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos \varphi & -\rho \sin \varphi \\ \sin \varphi & \rho \cos \varphi \end{vmatrix} = \rho.$$

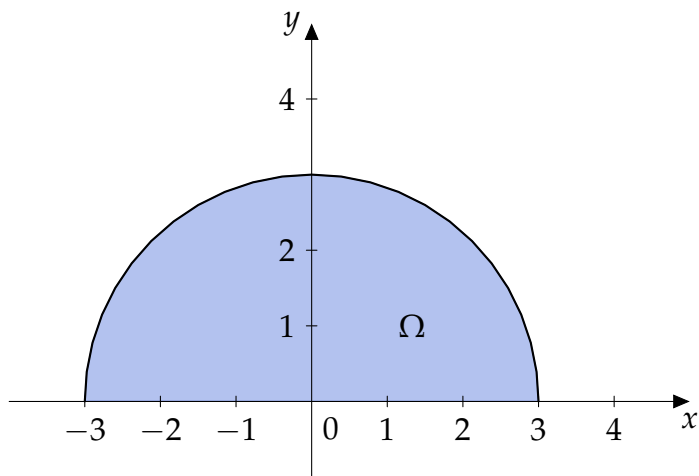
The transformation of the double integral to polar coordinates can be written in the form

$$\iint_{\Omega} f(x, y) \, dx \, dy = \iint_{\Omega^*} f(\rho \cos \varphi, \rho \sin \varphi) \rho \, d\rho \, d\varphi.$$

## 26 – Double integral in polar coordinates

## Example

Compute  $\iint_{\Omega} y \, dx \, dy$  over the domain  $\Omega = \{[x, y] : x^2 + y^2 \leq 9, y \geq 0\}$  using transformation to polar coordinates.



The domain  $\Omega$  is an upper half of the disc with the center in the origin of coordinates and with radius  $r = 3$ . We use transformation to polar coordinates and obtain the domain

$$\Omega^*: \quad 0 < \rho \leq 3, \\ 0 \leq \varphi \leq \pi.$$

We have

$$\begin{aligned} \iint_{\Omega} y \, dx \, dy &= \iint_{\Omega^*} \rho \sin \varphi \cdot \rho \, d\rho \, d\varphi = \int_0^3 \rho^2 \, d\rho \cdot \int_0^{\pi} \sin \varphi \, d\varphi \\ &= \left[ \frac{\rho^3}{3} \right]_0^3 \cdot [-\cos \varphi]_0^{\pi} = 18. \end{aligned}$$

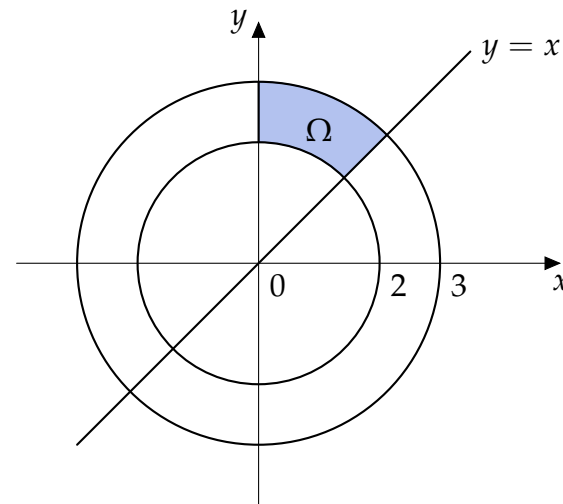
## Example

Compute  $\iint_{\Omega} x \, dx \, dy$  over the domain  $\Omega = \{[x, y] : 4 \leq x^2 + y^2 \leq 9, y \geq x, x \geq 0\}$ .

We can see real advantage of the transformation on this domain. While using Cartesian coordinates would be complicated, domain

$$\Omega^* = \left\{ [\rho, \varphi] : \rho \in [2, 3], \varphi \in \left[ \frac{\pi}{4}, \frac{\pi}{2} \right] \right\}$$

for polar coordinates is rectangular.

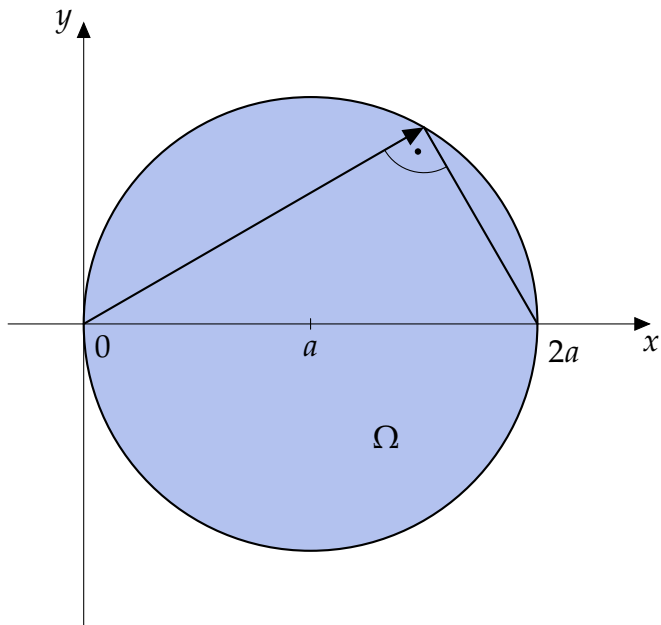


$$\begin{aligned} \iint_{\Omega} x \, dx \, dy &= \iint_{\Omega^*} \rho \cos \varphi \cdot \rho \, d\rho \, d\varphi = \int_2^3 \rho^2 \, d\rho \cdot \int_{\pi/4}^{\pi/2} \cos \varphi \, d\varphi \\ &= \left[ \frac{\rho^3}{3} \right]_2^3 \cdot [\sin \varphi]_{\pi/4}^{\pi/2} = \frac{19}{2} \left( 1 - \frac{\sqrt{2}}{2} \right) \end{aligned}$$

## 27 – Double integral in polar coordinates

**Example**

Calculate limits of the integral transformed to polar coordinates for the domain  $\Omega = \{[x, y]: x^2 + y^2 \leq 2ax\}$ .



First, we find the center and radius of the disc.

$$\begin{aligned}x^2 + y^2 &\leq 2ax \\x^2 - 2ax + a^2 + y^2 &\leq a^2 \\(x - a)^2 + y^2 &\leq a^2\end{aligned}$$

We have found that center  $S = [a, 0]$  and radius  $r = a$ .

**Remark**

Although, limits of  $\varphi$  are usually between  $0 \leq \varphi \leq 2\pi$ , in such cases, we use negative limits, to prevent splitting of the domain.

The azimuth must fulfil  $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$ . We can see that the upper limit of coordinate  $\rho$  depends on the azimuth  $\varphi$ . We obtain the value of the limit by substituting transformation equations to boundary equations of  $\Omega$ .

$$\begin{aligned}x^2 + y^2 &= 2ax \\ \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi &= 2a\rho \cos \varphi \\ \rho^2 &= 2a\rho \cos \varphi \\ \rho(\rho - 2a \cos \varphi) &= 0\end{aligned}$$

Roots  $\rho_1 = 0$  and  $\rho_2 = 2a \cos \varphi$  are limits of the integral. However, it is necessary to realise the dependency of coordinate  $\rho$  on coordinate  $\varphi$ . We can't calculate integrals over such domains as in case of rectangular ones. We need to use Fubini's theorem. The integral of an arbitrary function can be written as

$$\iint_{\Omega} f(x, y) \, dx \, dy = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a \cos \varphi} f(\rho \cos \varphi, \rho \sin \varphi) \rho \, d\rho.$$

## 28 – Double integral in polar coordinates

## Exercise

Compute following integrals over their domains  $\Omega$ :

a)  $\iint_{\Omega} (1 - 2x - 3y) \, dx \, dy, \quad \Omega: x^2 + y^2 \leq 2,$

b)  $\iint_{\Omega} \sqrt{1 - x^2 - y^2} \, dx \, dy, \quad \Omega: x^2 + y^2 \leq 1, x \geq 0, y \geq 0.$

## 29 – Double integral in polar coordinates

### Exercise

Compute integral  $\iint_{\Omega} \sin \sqrt{x^2 + y^2} \, dx \, dy$  over domain  $\Omega: \pi^2 \leq x^2 + y^2 \leq 4\pi^2$ .

## 30 – Double integral in polar coordinates

### Exercise

Compute integral  $\iint_{\Omega} \frac{\ln(x^2 + y^2)}{x^2 + y^2} dx dy$  over domain  $\Omega: 1 \leq x^2 + y^2 \leq e$ .

## 31 – Double integral in polar coordinates

### Exercise

Compute integral  $\iint_{\Omega} \sqrt{4 - x^2 - y^2} \, dx \, dy$  over domain  $\Omega: x^2 + y^2 \leq 2x$ .

## 32 – Double integral in polar coordinates

### Exercise

Compute integral  $\iint_{\Omega} xy \, dx \, dy$  over domain  $\Omega: x^2 + y^2 \leq 4y, y \geq x \geq 0$ .

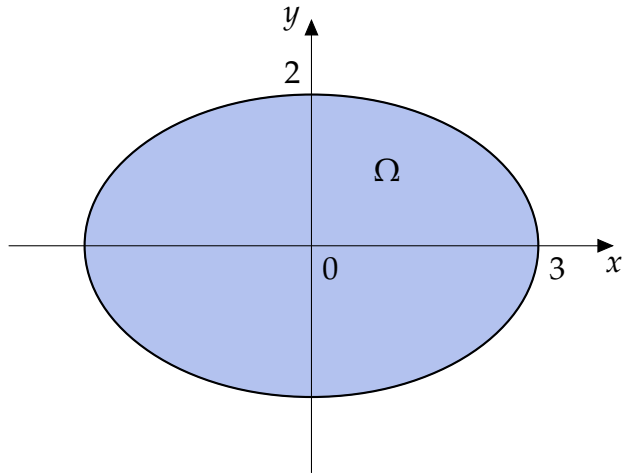


## 33 – Double integral in generalized polar coordinates

## Example

Compute  $\iint_{\Omega} \sqrt{4 - \frac{x^2}{9} - \frac{y^2}{4}} dx dy$  over  $\Omega = \{[x, y] : 4x^2 + 9y^2 \leq 36\}$  using transformation to generalized polar coordinates.

The boundary of the domain can be written in the form  $\frac{x^2}{9} + \frac{y^2}{4} = 1$ . Therefore, the domain is ellipse with center in the origin of coordinates and semi-axis  $a = 3$ ,  $b = 2$ .



In such case we use **generalized polar coordinates** in the form

$$\begin{aligned}x &= a\rho \cos \varphi, \\y &= b\rho \sin \varphi.\end{aligned}$$

For Jacobian of the transformation we obtain

$$J(\rho, \varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a \cos \varphi & -a\rho \sin \varphi \\ b \sin \varphi & b\rho \cos \varphi \end{vmatrix} = ab\rho.$$

Using generalized polar coordinates we obtained transformed domain

$$\Omega^* = \{[\rho, \varphi] : \rho \in (0, 1], \varphi \in [0, 2\pi)\}$$

and we can solve the integral now.

$$\begin{aligned}\iint_{\Omega} \sqrt{4 - \frac{x^2}{9} - \frac{y^2}{4}} dx dy &= \iint_{\Omega^*} \sqrt{4 - \frac{(3\rho \cos \varphi)^2}{9} - \frac{(2\rho \sin \varphi)^2}{4}} 6\rho d\rho d\varphi \\ &= 6 \iint_{\Omega^*} \sqrt{4 - \rho^2} \rho d\rho d\varphi = 6 \int_0^{2\pi} d\varphi \cdot \int_0^1 \sqrt{4 - \rho^2} \rho d\rho = 6 \cdot 2\pi \cdot \frac{1}{3} (8 - 3\sqrt{3}) \\ &= 4\pi (8 - 3\sqrt{3})\end{aligned}$$

Integral over coordinate  $\rho$  was calculated using substitution

$$\begin{aligned}4 - \rho^2 &= t, \\ -2\rho d\rho &= dt.\end{aligned}$$

## 34 – Double integral in generalized polar coordinates

### Exercise

Compute integral  $\iint_{\Omega} (2x + y) \, dx \, dy$  over domain  $\Omega: 4x^2 + y^2 \leq 16, y \leq 0, x \leq 0$ .

## 35 – Double integral in generalized polar coordinates

### Exercise

Compute integral  $\iint_{\Omega} xy \, dx \, dy$  over domain  $\Omega: x^2 + 4y^2 \leq 4, x \geq 0, y \geq 0$ .

## 36 – Practical applications of the double integral, area of a region

The area of a region  $\Omega$  is given by

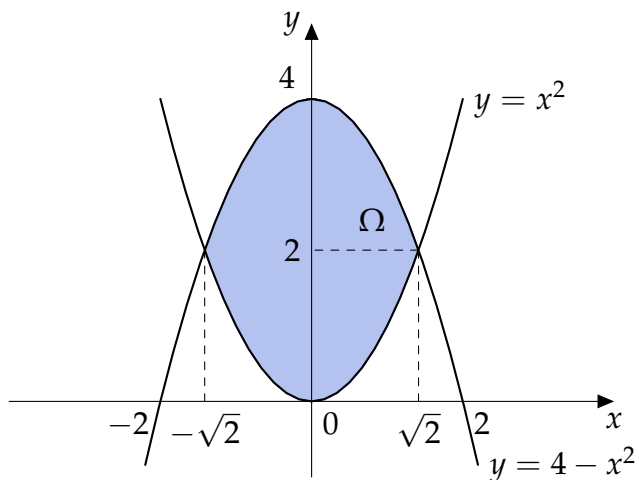
$$A = \iint_{\Omega} dx dy.$$

**Example**

Calculate the area of a region  $\Omega$  bounded by curves  $y = x^2$ ,  $y = 4 - x^2$ .

We need to find intersections of both parabolas  $y = x^2$ ,  $y = 4 - x^2$  that are  $x = \pm\sqrt{2}$ . We write the domain as a normal with respect to the  $x$ -axis with inequalities in the form:

$$\Omega: \quad -\sqrt{2} \leq x \leq \sqrt{2}, \\ x^2 \leq y \leq 4 - x^2.$$



We compute the area of our region using symmetry of the domain with respect to the  $y$ -axis

$$A = \iint_{\Omega} dx dy = \int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{x^2}^{4-x^2} dy \\ = 2 \int_0^{\sqrt{2}} (4 - 2x^2) dx = 2 \left[ 4x - \frac{2}{3}x^3 \right]_0^{\sqrt{2}} = \frac{16\sqrt{2}}{3}.$$

## 37 – Practical applications of the double integral, area of a region

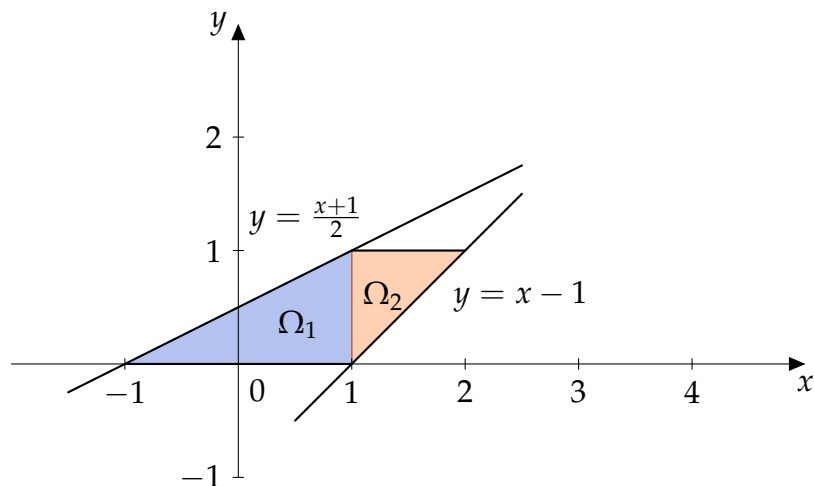
**Example**

Compute the area of domain

$$\Omega = \{[x, y]: x - y - 1 \leq 0, x - 2y + 1 \geq 0, 0 \leq y \leq 1\}.$$

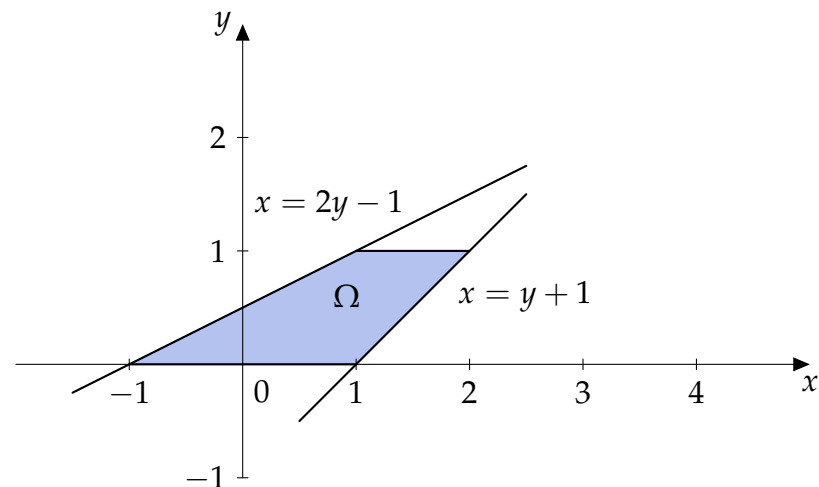
Domain is bounded by the lines  $y = 0$ ,  $y = 1$ ,  $y = x - 1$  and  $y = \frac{x+1}{2}$ .

If we write the domain as a normal with respect to the  $x$ -axis, we have to split the domain. It is a good exercise to compute the example in this way.



We write the domain as a normal with respect to the  $y$ -axis with inequalities in the form:

$$\Omega: \quad 2y - 1 \leq x \leq y + 1, \\ 0 \leq y \leq 1$$



and compute the area of  $\Omega$ :

$$A = \iint_{\Omega} dx dy = \int_0^1 dx \int_{2y-1}^{y+1} dy = \int_0^1 (2-y) dy = \left[ 2y - \frac{y^2}{2} \right]_0^1 = \frac{3}{2}.$$

## 38 – Practical applications of the double integral, area of a region

### Exercise

Compute the areas of the regions bounded by curves:

a)  $y = x, y = 5x, x = 1,$

b)  $y = x^2, y = 4 - x^2.$

## 39 – Practical applications of the double integral, area of a region

### Exercise

Compute the area of the region bounded by curves  $y = 2^x$ ,  $y = 2^{-2x}$ ,  $y = 4$ .

## 40 – Practical applications of the double integral, area of a region

### Exercise

Compute the area of the region bounded by curves  $x^2 + y^2 = 4$ ,  $x^2 + y^2 = 4y$ .

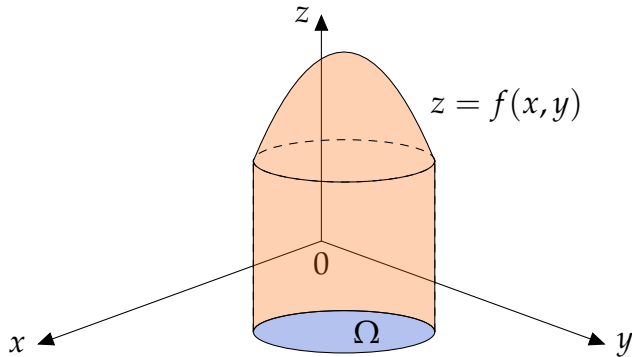


## 41 – Practical applications of the double integral, volume of a body

We know that  $\iint_D f(x, y) \, dx \, dy$  of a positive function  $f(x, y) > 0$  over a rectangular domain  $D$  has a meaning of the volume of the prism with a rectangular base  $D$  bounded from above by the function  $f(x, y)$ . If we replace rectangle  $D$  by general domain  $\Omega$ , we obtain  $\iint_{\Omega} f(x, y) \, dx \, dy$  and we calculate volume of the cylindrical body with basis  $\Omega$  and bounded from above by the function  $f(x, y)$ .

The **volume of the cylindrical body with basis  $\Omega$**  bounded by an arbitrary function  $f(x, y)$  is given by

$$V = \iint_{\Omega} |f(x, y)| \, dx \, dy.$$

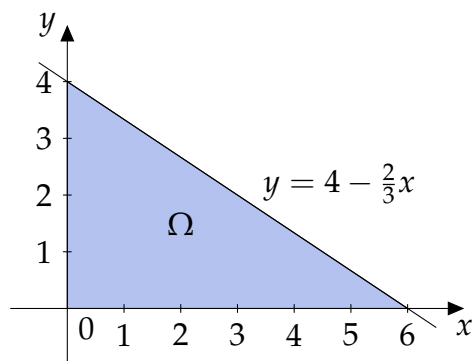


## 42 – Practical applications of the double integral, volume of a body

**Example**

Calculate the volume of the body bounded by surfaces  $2x + 3y = 12$ ,  $2z = y^2$ ,  $x = 0$ ,  $y = 0$ ,  $z = 0$ .

The basis of the body lies in the plane  $z = 0$ . Planes  $2x + 3y = 12$ ,  $x = 0$  and  $y = 0$  are perpendicular to the basis, thus they define the triangular domain  $\Omega$ .



Because  $z = \frac{y^2}{2} \geq 0$  for all  $[x, y] \in \Omega$ , therefore the surface  $z = \frac{y^2}{2}$  bounds the body from above. We write the domain as a normal with respect to the  $x$ -axis with inequalities for  $\Omega$  in the form:

$$\Omega: \quad 0 \leq x \leq 6, \\ 0 \leq y \leq 4 - \frac{2}{3}x.$$

We compute the volume of the body

$$\begin{aligned} V &= \iint_{\Omega} \frac{y^2}{2} dx dy = \frac{1}{2} \int_0^6 dx \int_0^{4-\frac{2}{3}x} y^2 dy = \frac{1}{2} \int_0^6 \left[ \frac{y^3}{3} \right]_0^{4-\frac{2}{3}x} dx \\ &= \frac{1}{6} \int_0^6 \left( 4 - \frac{2}{3}x \right)^3 dx = \frac{1}{6} \cdot \left( -\frac{3}{2} \right) \left[ \frac{\left( 4 - \frac{2}{3}x \right)^4}{4} \right]_0^6 = 16. \end{aligned}$$

## 43 – Practical applications of the double integral, volume of a body

### Exercise

Compute the volume of the body bounded by surfaces  $x = 0$ ,  $y = 0$ ,  $z = 0$ ,  $6x + 3y + z - 12 = 0$ .

## 44 – Practical applications of the double integral, volume of a body

### Exercise

Compute the volume of the body bounded by surfaces  $z = 0$ ,  $z = xy$ ,  $y = 0$ ,  $y = \sqrt{x}$ ,  $x + y = 2$ .

## 45 – Practical applications of the double integral, volume of a body

### Exercise

Compute the volume of the body bounded by surfaces  $z = 0$ ,  $2y = x^2$ ,  $z = y^2 - 4$ .

## 46 – Practical applications of the double integral, volume of a body

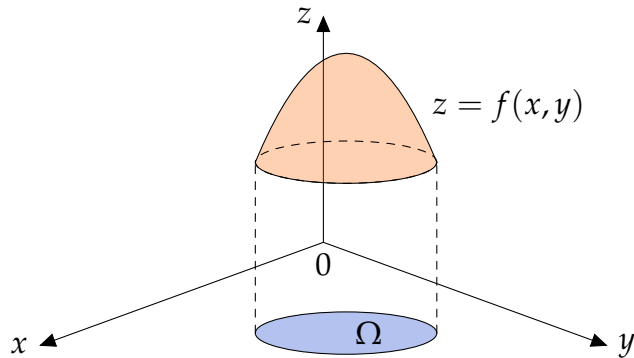
### Exercise

Compute the volume of the body bounded by surfaces  $z = 0$ ,  $z = 1 - x^2 - y^2$ .

## 47 – Practical applications of the double integral, surface area

We are able to compute the **area of the surface**  $z = f(x, y)$  where  $[x, y]$  is a point in the region  $\Omega$ . Function  $z = f(x, y)$  must have continuous partial derivatives on  $\Omega$ . In this case surface area is given by

$$S = \iint_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy.$$



## 48 – Practical applications of the double integral, surface area

**Example**

Calculate the area of a surface  $z = \sqrt{2xy}$  bounded by planes  $x = 1$ ,  $x = 2$ ,  $y = 1$  and  $y = 4$ .

Partial derivatives of  $z$  are  $\frac{\partial z}{\partial x} = \frac{y}{\sqrt{2xy}}$  and  $\frac{\partial z}{\partial y} = \frac{x}{\sqrt{2xy}}$ . The domain  $\Omega$  is a rectangle given by inequalities

$$\Omega: \quad 1 \leq x \leq 2, \\ 1 \leq y \leq 4$$

and we calculate the surface area

$$\begin{aligned} S &= \iint_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy = \iint_{\Omega} \sqrt{1 + \frac{y^2}{2xy} + \frac{x^2}{2xy}} dx dy = \iint_{\Omega} \sqrt{\frac{2xy + x^2 + y^2}{2xy}} dx dy = \iint_{\Omega} \sqrt{\frac{(x+y)^2}{2xy}} dx dy = \iint_{\Omega} \frac{x+y}{\sqrt{2xy}} dx dy \\ &= \int_1^2 dx \int_1^4 \left( \sqrt{\frac{x}{2}} \cdot y^{-\frac{1}{2}} + \frac{1}{\sqrt{2x}} \cdot y^{\frac{1}{2}} \right) dy = \int_1^2 \left[ \sqrt{\frac{x}{2}} \cdot 2y^{\frac{1}{2}} + \frac{1}{\sqrt{2x}} \cdot \frac{2}{3} y^{\frac{3}{2}} \right]_1^4 dx = \int_1^2 \left( \sqrt{\frac{x}{2}} \cdot 4 + \frac{1}{\sqrt{2x}} \cdot \frac{16}{3} - \sqrt{\frac{x}{2}} \cdot 2 - \frac{1}{\sqrt{2x}} \cdot \frac{2}{3} \right) dx \\ &= \int_1^2 \left( \sqrt{\frac{x}{2}} \cdot 2 + \frac{1}{\sqrt{2x}} \cdot \frac{14}{3} \right) dx = \int_1^2 \left( \frac{2}{\sqrt{2}} \cdot x^{\frac{1}{2}} + \frac{14}{3\sqrt{2}} \cdot x^{-\frac{1}{2}} \right) dx = \left[ \frac{2}{\sqrt{2}} \cdot \frac{2}{3} x^{\frac{3}{2}} + \frac{14}{3\sqrt{2}} \cdot 2x^{\frac{1}{2}} \right]_1^2 = \frac{8}{3} + \frac{28}{3} - \frac{4}{3\sqrt{2}} - \frac{28}{3\sqrt{2}} = 12 - \frac{32}{3\sqrt{2}} = 12 - \frac{16}{3}\sqrt{2}. \end{aligned}$$



## 49 – Practical applications of the double integral, surface area

### Exercise

Compute the area of the surface  $x + y + z = 4$  bounded by planes  $x = 0$ ,  $x = 2$ ,  $y = 0$ ,  $y = 2$ .

## 50 – Practical applications of the double integral, surface area

### Exercise

Compute the area of the surface  $y^2 + z^2 = 9$  bounded by planes  $x = 0$ ,  $x = 2$ ,  $y = -3$ ,  $y = 3$ .

## 51 – Practical applications of the double integral, surface area

### Exercise

Compute the area of the surface  $z = xy$  in the cylinder  $x^2 + y^2 = 4$ .

## 52 – Practical applications of the double integral, center of mass

Let  $\sigma(x, y) > 0$  be a surface density defined for each  $[x, y] \in \Omega$ . The **mass** of a domain  $\Omega$  is defined by

$$m = \iint_{\Omega} \sigma(x, y) \, dx \, dy.$$

**Static moment** of the domain  $\Omega$  with respect to the  $x$ -axis resp.  $y$ -axis is given by

$$S_x = \iint_{\Omega} y \sigma(x, y) \, dx \, dy \quad \text{resp.} \quad S_y = \iint_{\Omega} x \sigma(x, y) \, dx \, dy.$$

The coordinates of the **center of mass**  $C = [\xi, \eta]$  can be expressed as

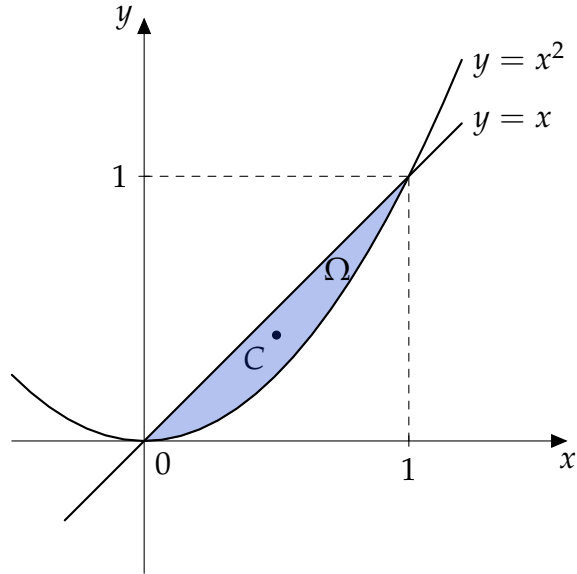
$$\xi = \frac{S_y}{m}, \quad \eta = \frac{S_x}{m}.$$

## 53 – Practical applications of the double integral, center of mass

**Example**

Compute the coordinates of the center of mass of the homogeneous region bounded by curves  $y = x$  and  $y = x^2$ .

The curves have intersections in points  $[0,0]$  and  $[1,1]$ . See figure.



We write the domain as a normal with respect to the  $x$ -axis with inequalities in the form:

$$\Omega: \quad 0 \leq x \leq 1, \\ x^2 \leq y \leq x.$$

First we compute the mass of the homogeneous domain

$$m = \iint_{\Omega} \sigma(x, y) \, dx \, dy = \sigma \int_0^1 dx \int_{x^2}^x dy$$

$$= \sigma \int_0^1 (x - x^2) \, dx = \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Then we calculate static moments

$$S_x = \iint_{\Omega} y \sigma(x, y) \, dx \, dy = \sigma \int_0^1 dx \int_{x^2}^x y \, dy = \sigma \int_0^1 \left[ \frac{y^2}{2} \right]_{x^2}^x dx$$

$$= \sigma \int_0^1 \left( \frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[ \frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15}.$$

$$S_y = \iint_{\Omega} x \sigma(x, y) \, dx \, dy = \sigma \int_0^1 dx \int_{x^2}^x x \, dy = \sigma \int_0^1 [xy]_{x^2}^x dx$$

$$= \sigma \int_0^1 (x^2 - x^3) \, dx = \left[ \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

Coordinates of the center are

$$\xi = \frac{S_y}{m} = \frac{1}{2}, \quad \eta = \frac{S_x}{m} = \frac{2}{5}$$

and center of mass is

$$C = [\xi, \eta] = \left[ \frac{1}{2}, \frac{2}{5} \right].$$

## 54 – Practical applications of the double integral, center of mass

### Exercise

Compute the center of mass coordinates of the homogeneous region bounded by curves  $y = x^2$ ,  $x = 4$ ,  $y = 0$ .

# **Worksheets for Mathematics III**

Triple integral

## 56 – Triple integral over rectangular hexahedron

While we use a double integral to integrate over a two-dimensional regions, we similarly use triple integral to integrate over three-dimensional regions. The definition of the three-dimensional integral over a rectangular hexahedron is similar to definition we used for double integral over a rectangle.

Let  $u = f(x, y, z)$  is a function of three variables that is continuous and bounded on the rectangular hexahedron

$$G = \{[x, y, z] \in \mathbb{R}^3 : x \in [a, b], y \in [c, d], z \in [e, h]\}.$$

We divide intervals  $[a, b]$ ,  $[c, d]$ ,  $[e, h]$  by three sequences of points

$$a = x_0 < x_1 < x_2 < \dots < x_m = b,$$

$$c = y_0 < y_1 < y_2 < \dots < y_n = d$$

and

$$e = z_0 < z_1 < z_2 < \dots < z_p = h$$

to intervals  $[x_{i-1}, x_i]$ ,  $i = 1, 2, \dots, m$ ,  $[y_{j-1}, y_j]$ ,  $j = 1, 2, \dots, n$  and  $[z_{k-1}, z_k]$ ,  $k = 1, 2, \dots, p$ . We denote  $\Delta x_i = x_i - x_{i-1}$ ,  $\Delta y_j = y_j - y_{j-1}$  and  $\Delta z_k = z_k - z_{k-1}$ .

The planes that lead through points  $x_i$  or  $y_j$  or  $z_k$  parallel to coordinate planes divide the hexahedron  $G$  to  $m \cdot n \cdot p$  small hexahedrons  $G_{ijk}$  (see Figure) with volume of each  $\Delta G_{ijk} = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$ . We choose an arbitrary point  $[\xi_i, \eta_j, \zeta_k]$  in each hexahedron  $G_{ijk}$  and we create products  $f(\xi_i, \eta_j, \zeta_k) \cdot \Delta G_{ijk} = f(\xi_i, \eta_j, \zeta_k) \cdot \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$  that for positive function  $f(x, y, z) \geq 0$  has a physical meaning of the mass of the hexahedron  $G_{ijk}$  with density  $f(\xi_i, \eta_j, \zeta_k)$ . The sum of these products

$$\sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(\xi_i, \eta_j, \zeta_k) \cdot \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

represents the mass of the body consisted of such hexahedrons.

**Definition**

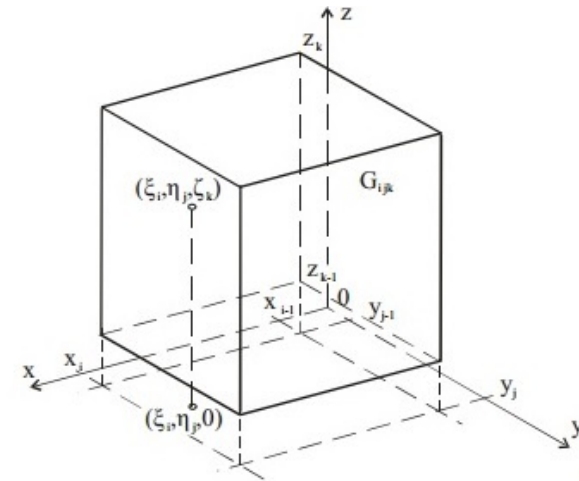
If there exists

$$\lim \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^p f(\xi_i, \eta_j, \zeta_k) \cdot \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

for  $m \rightarrow \infty$ ,  $n \rightarrow \infty$ ,  $p \rightarrow \infty$ ,  $\Delta x_i \rightarrow 0$ ,  $\Delta y_j \rightarrow 0$ ,  $\Delta z_k \rightarrow 0$  for all  $i = 1, 2, \dots, m$ ,  $j = 1, 2, \dots, n$ ,  $k = 1, 2, \dots, p$ , we call it a **triple integral** of function  $f(x, y, z)$  over the rectangular hexahedron  $G$  and denote it by

$$\iiint_G f(x, y, z) dx dy dz.$$

The triple integral over a hexahedron  $G$  of a positive function  $f(x, y, z) > 0$  has a meaning of the mass of a hexahedron  $G$  with density  $f(x, y, z)$ .





## 57 – Triple integral over rectangular hexahedron

**Theorem (Fubini's theorem)**

Let  $G = \{[x, y, z] \in \mathbb{R}^3 : x \in [a, b], y \in [c, d], z \in [e, h]\}$ . If a function  $f(x, y, z)$  is continuous on the hexahedron  $G$ , then

$$\iiint_G f(x, y, z) \, dx \, dy \, dz = \int_a^b \left( \int_c^d \left( \int_e^h f(x, y, z) \, dz \right) dy \right) dx.$$

The Theorem is similar to two-dimensional Fubini's theorem. We can rewrite the formula by using a different order of integration in five more ways. The triple integral is then converted to three one-dimensional integrals. Similarly to the double integral, we can write

$$\int_a^b \left( \int_c^d \left( \int_e^h f(x, y, z) \, dz \right) dy \right) dx = \int_a^b dx \int_c^d dy \int_e^h f(x, y, z) \, dz.$$

If the integrand  $f(x, y, z)$  can be written as a product of three functions of one variable  $f(x, y, z) = f_1(x) \cdot f_2(y) \cdot f_3(z)$ , it holds:

$$\iiint_G f(x, y, z) \, dx \, dy \, dz = \int_a^b f_1(x) \, dx \cdot \int_c^d f_2(y) \, dy \cdot \int_e^h f_3(z) \, dz.$$

## 58 – Triple integral over rectangular hexahedron

**Theorem (Properties of the triple integral over a rectangular hexahedron)**

$$1. \iiint_G cf(x, y, z) \, dx \, dy \, dz = c \iiint_G f(x, y, z) \, dx \, dy \, dz,$$

$$2. \iiint_G (f(x, y, z) + g(x, y, z)) \, dx \, dy \, dz = \iiint_G f(x, y, z) \, dx \, dy \, dz + \iiint_G g(x, y, z) \, dx \, dy \, dz,$$

$$3. \iiint_G f(x, y, z) \, dx \, dy \, dz = \iiint_{G_1} f(x, y, z) \, dx \, dy \, dz + \iiint_{G_2} f(x, y, z) \, dx \, dy \, dz,$$

where  $f, g$  are continuous functions on  $G$ ,  $c \in \mathbb{R}$  and  $G_1, G_2$  are non-overlapping hexahedrons that fulfil  $G = G_1 \cup G_2$ .

## 59 – Triple integral over rectangular hexahedron

**Example**

Compute  $\iiint_G xy^2z \, dx \, dy \, dz$  over the rectangular hexahedron

$$G = \{[x, y, z] \in \mathbb{R}^3 : x \in [0, 2], y \in [1, 3], z \in [1, 2]\}.$$

$$\begin{aligned} \iiint_G xy^2z \, dx \, dy \, dz &= \int_0^2 x \, dx \cdot \int_1^3 y^2 \, dy \cdot \int_1^2 z \, dz \\ &= \left[\frac{x^2}{2}\right]_0^2 \cdot \left[\frac{y^3}{3}\right]_1^3 \cdot \left[\frac{z^2}{2}\right]_1^2 = 2 \cdot \left(9 - \frac{1}{3}\right) \cdot \left(2 - \frac{1}{2}\right) \\ &= 2 \cdot \frac{26}{3} \cdot \frac{3}{2} = 26 \end{aligned}$$

**Example**

Compute  $\iiint_G (x + y) \, dx \, dy \, dz$  over the rectangular hexahedron

$$G = \{[x, y, z] \in \mathbb{R}^3 : x \in [0, 1], y \in [0, 2], z \in [0, 3]\}.$$

$$\begin{aligned} \iiint_G (x + y) \, dx \, dy \, dz &= \int_0^1 dx \int_0^2 dy \int_0^3 (x + y) \, dz = \int_0^1 dx \int_0^2 (x + y) [z]_0^3 \, dy \\ &= 3 \int_0^1 dx \int_0^2 (x + y) \, dy = 3 \int_0^1 \left[xy + \frac{y^2}{2}\right]_0^2 \, dx = 3 \int_0^1 (2x + 2) \, dx \\ &= 6 \left[\frac{x^2}{2} + x\right]_0^1 = 6 \cdot \frac{3}{2} = 9 \end{aligned}$$

**Remark**

If it is not possible to decompose the integrand as a product of three one-dimensional integrals we can always use Fubini's theorem.

## 60 – Triple integral over rectangular hexahedron

## Exercise

Compute following integrals over their domains  $G$ .

$$\text{a) } \iiint_G xy^2 \sqrt{z} \, dx \, dy \, dz, \quad G = \{[x, y, z] \in \mathbb{R}^3 : x \in [-2, 1], y \in [1, 3], z \in [2, 4]\}$$

$$\text{b) } \iiint_G \frac{1}{1-x-y} \, dx \, dy \, dz, \quad G = \{[x, y, z] \in \mathbb{R}^3 : x \in [0, 1], y \in [2, 5], z \in [2, 4]\}$$

## 61 – Triple integral over rectangular hexahedron

## Exercise

Compute following integrals over their domains  $G$ .

$$\text{a) } \iiint_G \ln x^{yz} \, dx \, dy \, dz, \quad G = \{[x, y, z] \in \mathbb{R}^3 : x \in [1, 2], y \in [0, 1], z \in [0, 2]\}$$

$$\text{b) } \iiint_G \left( \frac{1}{x} + \frac{2}{y} + \frac{3}{z} \right) \, dx \, dy \, dz, \quad G = \{[x, y, z] \in \mathbb{R}^3 : x \in [1, 2], y \in [1, 2], z \in [1, 2]\}$$

## 62 – Triple integral over rectangular hexahedron

## Exercise

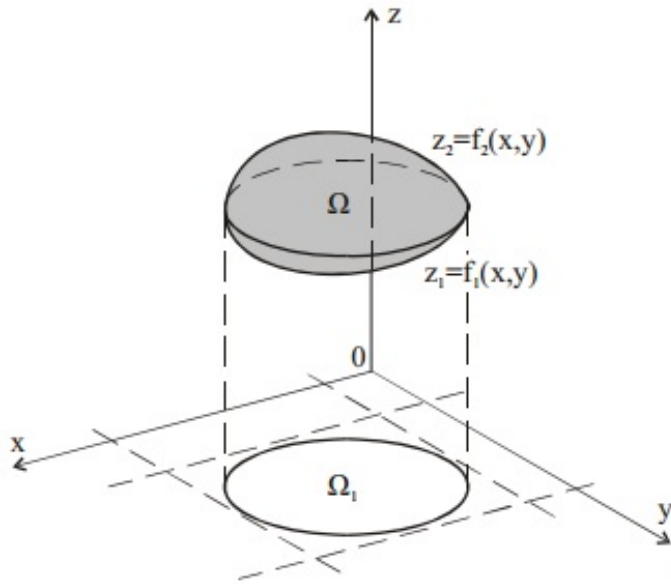
Compute following integrals over their domains  $G$ .

$$\text{a) } \iiint_G e^{x+y+z} dx dy dz, \quad G = \{[x, y, z] \in \mathbb{R}^3: x \in [0, 1], y \in [0, 1], z \in [0, 1]\}$$

$$\text{b) } \iiint_G \sqrt{xyz} dx dy dz, \quad G = \{[x, y, z] \in \mathbb{R}^3: x \in [0, 1], y \in [0, 9], z \in [0, 16]\}$$

## 63 – Triple integral over a general domain

Similarly like in two-dimensional case, we are able to generalize our problem of solving triple integrals over any three-dimensional region  $\Omega$  that is bounded by a closed surface. We consider only such surfaces that don't intersect themselves and lines parallel with  $z$ -axis leading through an arbitrary inner point of the surface intersect with the surface in two points. Such domain will be called **normal domain with respect to the coordinate plane  $xy$** .



We create an orthogonal projection  $\Omega_1$  of the domain  $\Omega$  into  $xy$ -plane. A variable  $z$  must fulfil

$$f_1(x, y) \leq z \leq f_2(x, y).$$

The domain  $\Omega_1$  is either normal with respect to  $x$ -axis or  $y$ -axis and we describe it using approach from the Double integral section by inequalities  $x_1 \leq x \leq x_2$ ,  $g_1(x) \leq y \leq g_2(x)$  resp.  $y_1 \leq y \leq y_2$ ,  $h_1(y) \leq x \leq h_2(y)$ . If the function  $f(x, y, z)$  is continuous on  $\Omega$  we use a method similar to Fubini's theorem used in the Double integral section and express

$$\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz = \int_{x_1}^{x_2} dx \int_{g_1(x)}^{g_2(x)} dy \int_{f_1(x, y)}^{f_2(x, y)} f(x, y, z) \, dz$$

resp.

$$\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz = \int_{y_1}^{y_2} dy \int_{h_1(y)}^{h_2(y)} dx \int_{f_1(x, y)}^{f_2(x, y)} f(x, y, z) \, dz.$$

We start to integrate with respect to variable  $z$ , limits are functions of two variables  $x, y$ . After that we calculate a double integral over a regular domain  $\Omega_1$ .

We are able to use analogical approach and create an orthogonal projection of the domain  $\Omega$  into planes either  $xz$  or  $yz$ . That way we can use six different orders of integration for an arbitrary domain.

**Remark**

The triple integral over a general closed domain has analogical properties as the triple integral over a rectangular hexahedron.

## 64 – Triple integral over a general domain

## Example

Determine integration limits for integral  $\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz$  over the domain  $\Omega$  that is bounded by surfaces  $z = \frac{1}{2}(x^2 + y^2)$  and  $z = 4 - \frac{1}{2}(x^2 + y^2)$ .

Both surfaces are rotational paraboloids and based on the figure where we can see projection of the body into  $yz$ -plane

$$\frac{1}{2}(x^2 + y^2) \leq z \leq 4 - \frac{1}{2}(x^2 + y^2).$$

We obtain an equation of the intersection of both surfaces from the equation

$$\frac{1}{2}(x^2 + y^2) = 4 - \frac{1}{2}(x^2 + y^2)$$

which leads to

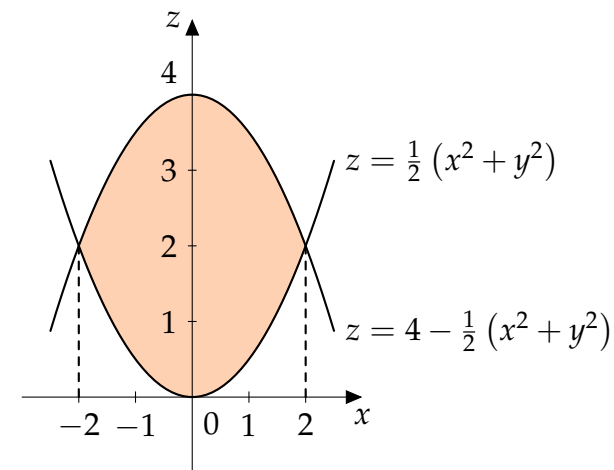
$$x^2 + y^2 = 4.$$

Therefore, the orthogonal projection  $\Omega_1$  of the domain  $\Omega$  to coordinate plane  $xy$  is a circle with the center in the origin of coordinates and radius  $r = 2$ . That domain can be treated as a normal domain with respect to the  $x$ -axis with inequalities

$$\begin{aligned} -2 &\leq x \leq 2, \\ -\sqrt{4 - x^2} &\leq y \leq \sqrt{4 - x^2}, \end{aligned}$$

as well as a normal domain with respect to the  $y$ -axis with inequalities

$$\begin{aligned} -2 &\leq y \leq 2, \\ -\sqrt{4 - y^2} &\leq x \leq \sqrt{4 - y^2}. \end{aligned}$$

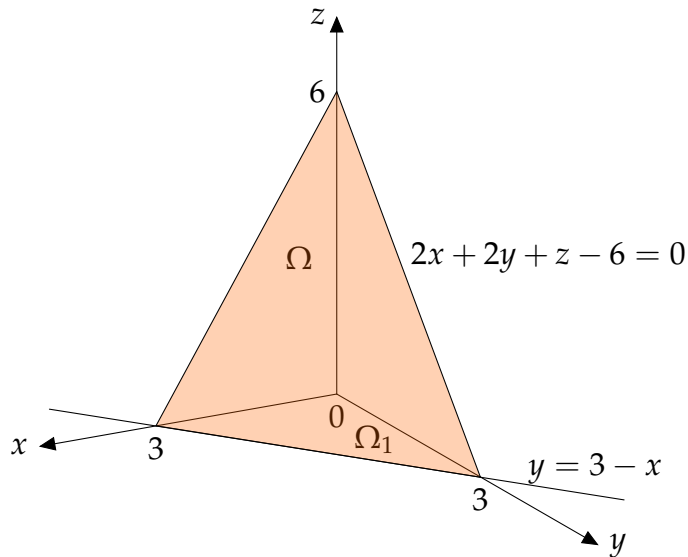




## 65 – Triple integral over a general domain

## Example

Compute integral  $\iiint_{\Omega} x \, dx \, dy \, dz$  over the domain  $\Omega$  bounded by surfaces  $x = 0, y = 0, z = 0, 2x + 2y + z - 6 = 0$ .



Based on the figure, the domain  $\Omega$  must fulfil

$$0 \leq z \leq 6 - 2x - 2y$$

The orthogonal projection  $\Omega_1$  of the domain  $\Omega$  to  $xy$ -plane is the triangle bounded by lines  $x = 0, y = 0, y = 3 - x$ . The last equation is intersection of planes  $2x + 2y + z - 6 = 0$  and  $z = 0$ .

We describe  $\Omega_1$  as normal with respect to  $x$ -axis by inequalities

$$\Omega_1: \quad 0 \leq x \leq 3, \\ 0 \leq y \leq 3 - x$$

and we can calculate the integral

$$\begin{aligned} \iiint_{\Omega} x \, dx \, dy \, dz &= \int_0^3 dx \int_0^{3-x} dy \int_0^{6-2x-2y} x \, dz = \int_0^3 dx \int_0^{3-x} [xz]_0^{6-2x-2y} dy \\ &= \int_0^3 dx \int_0^{3-x} x(6 - 2x - 2y) dy = \int_0^3 x [6y - 2xy - y^2]_0^{3-x} dx \\ &= \int_0^3 x [6(3-x) - 2x(3-x) - (3-x)^2] dx = \int_0^3 (x^3 - 6x^2 + 9x) dx \\ &= \left[ \frac{x^4}{4} - 2x^3 + \frac{9}{2}x^2 \right]_0^3 = \frac{27}{4}. \end{aligned}$$

## 66 – Triple integral over a general domain

### Exercise

Compute integral  $\iiint_{\Omega} x^3 y^2 z \, dx \, dy \, dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : y \geq 0, y \leq x, x \leq 1, z \geq 0, z \leq xy\}$ .

## 67 – Triple integral over a general domain

### Exercise

Compute integral  $\iiint_{\Omega} \frac{1}{1+x+y} dx dy dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}$ .

## 68 – Triple integral over a general domain

### Exercise

Compute integral  $\iiint_{\Omega} \frac{x+z}{4+y} dx dy dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3: x \geq 0, y \geq 0, z \geq 0, x+z \leq 3, y \leq 4\}$ .

## 69 – Triple integral over a general domain

### Exercise

Compute integral  $\iiint_{\Omega} y \cos(x+z) \, dx \, dy \, dz$  over domain  $\Omega = \left\{ [x, y, z] \in \mathbb{R}^3 : y \leq \sqrt{x}, y \geq 0, z \geq 0, x+z \leq \frac{\pi}{2} \right\}$ .

## 70 – Triple integral over a general domain

### Exercise

Compute integral  $\iiint_{\Omega} dx dy dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3: x + y \leq 1, y \geq 0, y \leq 2x, z \geq 0, z \leq 1 - x^2\}$ .

## 71 – Triple integral over a general domain

### Exercise

Compute integral  $\iiint_{\Omega} z \, dx \, dy \, dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : x^2 + y^2 \leq 4, 0 \leq z \leq 2, y \geq 0\}$ .

## 72 – Transformation of the triple integral

Similarly to the double integral, using Cartesian coordinates for some domains can be rather complicated. Especially in case of cylinders, cones or spheres. Therefore, we formulate analogical theorem that describes general transformation of the triple integral.

**Theorem (Transformation to general coordinates)**

- Let equations  $x = u(r, s, t), y = v(r, s, t), z = w(r, s, t)$  map the region  $\Omega$  bijectively to the region  $\Omega^*$ .
- Let function  $f(x, y, z)$  be continuous and bounded on  $\Omega$  and functions  $x = u(r, s, t), y = v(r, s, t), z = w(r, s, t)$  have continuous partial derivatives on  $\hat{\Omega}$  that fulfils  $\Omega^* \subset \hat{\Omega}$ .

- Let  $J(u, v, w) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{vmatrix} \neq 0$  in  $\Omega^*$ .

Then

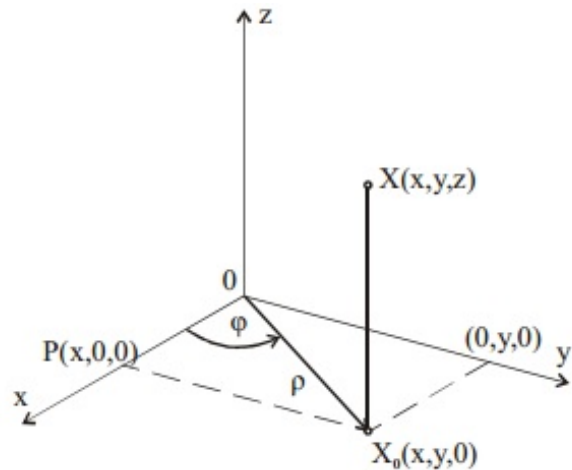
$$\iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz = \iiint_{\Omega^*} f(u(r, s, t), v(r, s, t), w(r, s, t)) |J| \, dr \, ds \, dt.$$

Determinant  $J(u, v, w) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{vmatrix}$  is again called **Jacobian** or **Jacobi determinant**.



## 73 – Transformation to cylindrical coordinates

**Transformation to cylindrical coordinates** is suitable for integration domains such as cylinders, cones or their parts. It is used in cases when orthogonal projection  $\Omega_1$  of the domain  $\Omega$  to plane  $xy$  is a disc or a part of a disc. We replace Cartesian coordinates  $x, y, z$  by cylindrical coordinates  $\rho, \varphi, z$ , according to the following figure.



The meaning of coordinates  $\rho, \varphi$  is the same as we have already used for polar coordinates and the third coordinate  $z$  doesn't change.

The transformation to cylindrical coordinates is given by transformation equations

$$x = \rho \cos \varphi,$$

$$y = \rho \sin \varphi,$$

$$z = z.$$

According to theorem describing transformation to general coordinates, we replace volume element  $dx dy dz$  by  $|J| d\rho d\varphi dz$ , where the Jacobian of the transformation to cylindrical coordinates satisfies

$$J(\rho, \varphi, z) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos \varphi & -\rho \sin \varphi & 0 \\ \sin \varphi & \rho \cos \varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

The transformation of the triple integral to cylindrical coordinates can then be written in the form

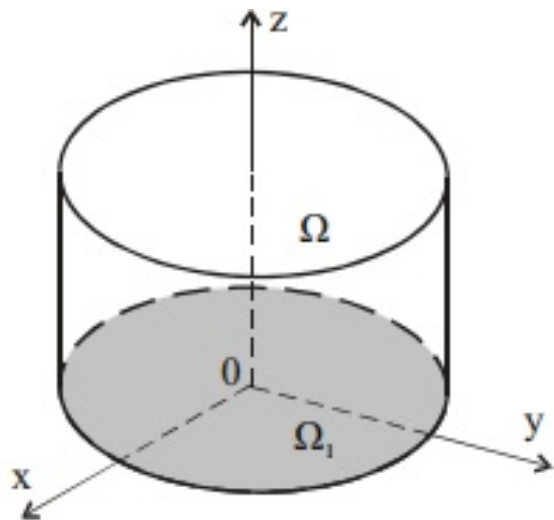
$$\iiint_{\Omega} f(x, y, z) dx dy dz = \iiint_{\Omega^*} f(\rho \cos \varphi, \rho \sin \varphi, z) \rho d\rho d\varphi dz.$$

## 74 – Transformation to cylindrical coordinates

**Example**

Compute  $\iiint_{\Omega} dx dy dz$  over the domain  $\Omega$  bounded by surfaces  
 $x^2 + y^2 = 1, z = 0, z = 1$ .

The domain  $\Omega$  is the rotational cylinder symmetrical with respect to the  $z$ -axis, with radius of the base  $\rho = 1$  and height  $z = 1$ , according to the following figure.



We need to determine the bounds of the transformed domain  $\Omega^*$ . It is obvious that  $0 \leq z \leq 1$ . Inequalities for coordinates  $\rho, \varphi$  are the same as for transformation to polar coordinates, i.e.  $0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi$ . Therefore

$$\begin{aligned} \iiint_{\Omega} dx dy dz &= \iiint_{\Omega^*} \rho d\rho d\varphi dz = \int_0^{2\pi} d\varphi \cdot \int_0^1 \rho d\rho \cdot \int_0^1 dz \\ &= [\varphi]_0^{2\pi} \cdot \left[ \frac{\rho^2}{2} \right]_0^1 \cdot [z]_0^1 = \pi. \end{aligned}$$

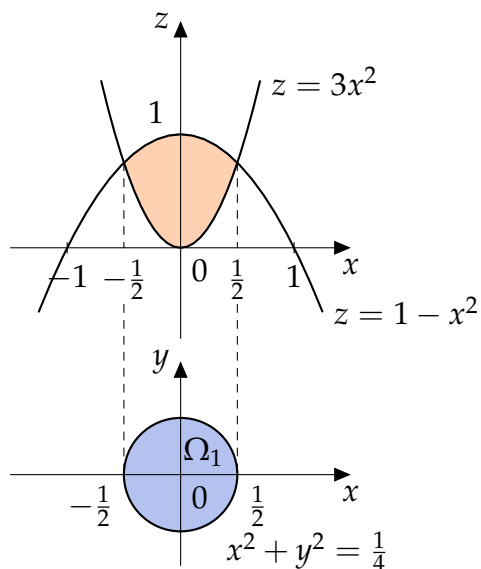
## 75 – Transformation to cylindrical coordinates

## Example

Compute  $\iiint_{\Omega} dx dy dz$  over the domain  $\Omega$  bounded by surfaces  $z = 3x^2 + 3y^2$ ,  $z = 1 - x^2 - y^2$ .

Both surfaces are paraboloids and the orthogonal projection  $\Omega_1$  of the domain  $\Omega$  to coordinate plane  $x, y$  is a ring, whose equation we obtain from the intersection of both paraboloids

$$\begin{aligned} 3x^2 + 3y^2 &= 1 - x^2 - y^2 \\ x^2 + y^2 &= \frac{1}{4}. \end{aligned}$$



Therefore, inequalities for coordinates  $\rho, \varphi$  must fulfil

$$\begin{aligned} 0 &\leq \rho \leq \frac{1}{2}, \\ 0 &\leq \varphi \leq 2\pi. \end{aligned}$$

We obtain limits of the variable  $z$  from equations of both paraboloids by substituting of variables

$$\begin{aligned} z &= 3x^2 + 3y^2 = 3\rho^2 \cos^2 \varphi + 3\rho^2 \sin^2 \varphi = 3\rho^2, \\ z &= 1 - x^2 - y^2 = 1 - \rho^2 \cos^2 \varphi - \rho^2 \sin^2 \varphi = 1 - \rho^2. \end{aligned}$$

Hence

$$3\rho^2 \leq z \leq 1 - \rho^2.$$

We compute the integral by using transformation to cylindrical coordinates

$$\begin{aligned} \iiint_{\Omega} dx dy dz &= \iiint_{\Omega^*} \rho d\rho d\varphi dz = \int_0^{2\pi} d\varphi \int_0^{1/2} d\rho \int_{3\rho^2}^{1-\rho^2} \rho dz \\ &= 2\pi \int_0^{1/2} \rho [z]_{3\rho^2}^{1-\rho^2} d\rho = 2\pi \int_0^{1/2} \rho (1 - 4\rho^2) d\rho = 2\pi \int_0^{1/2} (\rho - 4\rho^3) d\rho \\ &= 2\pi \left[ \frac{\rho^2}{2} - \rho^4 \right]_0^{1/2} = 2\pi \cdot \frac{1}{16} = \frac{\pi}{8}. \end{aligned}$$

## 76 – Transformation to cylindrical coordinates

### Exercise

Compute integral  $\iiint_{\Omega} dx dy dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3: x^2 + y^2 \leq 1, x \geq 0, 0 \leq z \leq 6\}$ .

## 77 – Transformation to cylindrical coordinates

### Exercise

Compute integral  $\iiint_{\Omega} z \, dx \, dy \, dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : x^2 + y^2 \leq 9, x \leq y \leq x\sqrt{3}, 0 \leq z \leq 4\}$ .

## 78 – Transformation to cylindrical coordinates

## Exercise

Compute integral  $\iiint_{\Omega} z \, dx \, dy \, dz$  over their domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}, z \leq 1\}$ .

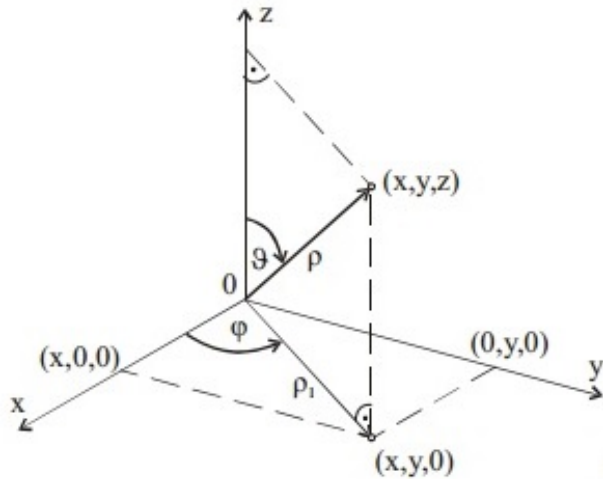
## 79 – Transformation to cylindrical coordinates

## Exercise

Compute integral  $\iiint_{\Omega} z\sqrt{x^2 + y^2} \, dx \, dy \, dz$  over their domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : x^2 + y^2 \leq 2x, 0 \leq z \leq 1\}$ .

## 80 – Transformation to spherical coordinates

Transformation to spherical coordinates is suitable for integrals where integration domains are spheres, ellipsoids or their parts. We replace Cartesian coordinates  $x, y, z$  by spherical coordinates  $\rho, \varphi, \vartheta$  according to the following figure.



The coordinate  $\rho$  denotes a distance between the point  $[x, y, z]$  and the origin of the coordinates,  $\varphi$  denotes positively oriented angle in coordinate  $xy$ -plane between positive part of the  $x$ -axis and the projection  $\rho_1$  of the radius vector  $\rho$  to coordinate  $xy$ -plane and  $\vartheta$  denotes positively oriented angle between positive part of the  $z$ -axis and the radius vector  $\rho$ .

We obtain transformation equations

$$\begin{aligned}x &= \rho \cos \varphi \sin \vartheta, \\y &= \rho \sin \varphi \sin \vartheta, \\z &= \rho \cos \vartheta.\end{aligned}$$

Jacobian of the transformation to spherical coordinates satisfies

$$\begin{aligned}J &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \vartheta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \end{vmatrix} = \begin{vmatrix} \cos \varphi \sin \vartheta & -\rho \sin \varphi \sin \vartheta & \rho \cos \varphi \cos \vartheta \\ \sin \varphi \sin \vartheta & \rho \cos \varphi \sin \vartheta & \rho \sin \varphi \cos \vartheta \\ \cos \vartheta & 0 & -\rho \sin \vartheta \end{vmatrix} \\ &= -\rho^2 \sin \vartheta.\end{aligned}$$

Transformation of the triple integral to spherical coordinates can be written in the form

$$\begin{aligned}& \iiint_{\Omega} f(x, y, z) \, dx \, dy \, dz \\ &= \iiint_{\Omega^*} f(\rho \cos \varphi \sin \vartheta, \rho \sin \varphi \sin \vartheta, \rho \cos \vartheta) \rho^2 \sin \vartheta \, d\rho \, d\varphi \, d\vartheta.\end{aligned}$$

The sphere with the center in the origin of coordinates and radius  $a$  transformed to spherical coordinates is mapped to domain  $\Omega^*$  given by inequalities

$$\begin{aligned}\Omega^*: \quad & 0 \leq \rho \leq a, \\ & 0 \leq \varphi \leq 2\pi, \\ & 0 \leq \vartheta \leq \pi.\end{aligned}$$



## 81 – Transformation to spherical coordinates

## Example

Compute  $\iiint_{\Omega} (x^2 + y^2 + z^2) \, dx \, dy \, dz$  over the domain  $\Omega$  bounded by  $x^2 + y^2 + z^2 \geq 1$  and  $x^2 + y^2 + z^2 \leq 4$ .

The domain  $\Omega$  is bounded by two spherical surfaces with the center in the origin of coordinates and radii  $\rho_1 = 1$ ,  $\rho_2 = 2$ . We use transformation to spherical coordinates with inequalities

$$\Omega^*: \quad \begin{aligned} 1 &\leq \rho \leq 2, \\ 0 &\leq \varphi \leq 2\pi, \\ 0 &\leq \vartheta \leq \pi. \end{aligned}$$

and calculate the integral using transformation to spherical coordinates

$$\begin{aligned} &\iiint_{\Omega} (x^2 + y^2 + z^2) \, dx \, dy \, dz \\ &= \iiint_{\Omega^*} (\rho^2 \cos^2 \varphi \sin^2 \vartheta + \rho^2 \sin^2 \varphi \sin^2 \vartheta + \rho^2 \cos^2 \vartheta) \rho^2 \sin \vartheta \, d\rho \, d\varphi \, d\vartheta \\ &= \int_1^2 d\rho \int_0^{2\pi} d\varphi \int_0^{\pi} \rho^4 \sin \vartheta \, d\vartheta = \int_1^2 \rho^4 \, d\rho \cdot \int_0^{2\pi} d\varphi \cdot \int_0^{\pi} \sin \vartheta \, d\vartheta \\ &= \left[ \frac{\rho^5}{5} \right]_1^2 \cdot [\varphi]_0^{2\pi} \cdot [-\cos \vartheta]_0^{\pi} = \frac{31}{5} \cdot 2\pi \cdot 2 = \frac{124}{5} \pi. \end{aligned}$$

## Example

Compute integral  $\iiint_{\Omega} z \, dx \, dy \, dz$  over the domain  $\Omega$  bounded by  $x^2 + y^2 + z^2 \leq 4$ ,  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$ .

The domain  $\Omega$  is one eighth of the sphere in the first octant with the center in the origin of coordinates and radius  $\rho = 2$ . Therefore

$$\Omega^*: \quad \begin{aligned} 0 &\leq \rho \leq 2, \\ 0 &\leq \varphi \leq \frac{\pi}{2}, \\ 0 &\leq \vartheta \leq \frac{\pi}{2}. \end{aligned}$$

and

$$\begin{aligned} &\iiint_{\Omega} z \, dx \, dy \, dz = \iiint_{\Omega^*} \rho \cos \vartheta \rho^2 \sin \vartheta \, d\rho \, d\varphi \, d\vartheta \\ &= \int_0^2 d\rho \int_0^{\pi/2} d\varphi \int_0^{\pi/2} \rho^3 \sin \vartheta \cos \vartheta \, d\vartheta = \int_0^2 \rho^3 \, d\rho \cdot \int_0^{\pi/2} d\varphi \cdot \int_0^{\pi/2} \frac{1}{2} \sin 2\vartheta \, d\vartheta \\ &= \left[ \frac{\rho^4}{4} \right]_0^2 \cdot [\varphi]_0^{\pi/2} \cdot \left[ -\frac{1}{4} \cos 2\vartheta \right]_0^{\pi/2} = 4 \cdot \frac{\pi}{2} \cdot \frac{1}{2} = \pi. \end{aligned}$$

## 82 – Transformation to spherical coordinates

### Exercise

Compute integral  $\iiint_{\Omega} z(x^2 + y^2) \, dx \, dy \, dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3: x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$ .

## 83 – Transformation to spherical coordinates

## Exercise

Compute integral  $\iiint_{\Omega} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x \geq 0, y \geq 0, z \geq 0\}$ .

## 84 – Transformation to spherical coordinates

### Exercise

Compute integral  $\iiint_{\Omega} (x^2 + y^2) \, dx \, dy \, dz$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : 4 \leq x^2 + y^2 + z^2 \leq 9, z \geq 0\}$ .

## 85 – Transformation to spherical coordinates

## Exercise

Compute integral  $\iiint_{\Omega} \frac{dx \, dy \, dz}{1 + x^2 + y^2 + z^2}$  over domain  $\Omega = \{[x, y, z] \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 4, x \leq y \leq x\sqrt{3}, z \geq 0\}$ .

## 86 – Practical applications of the triple integral, volume of a body

The volume of the body  $\Omega$  is given by

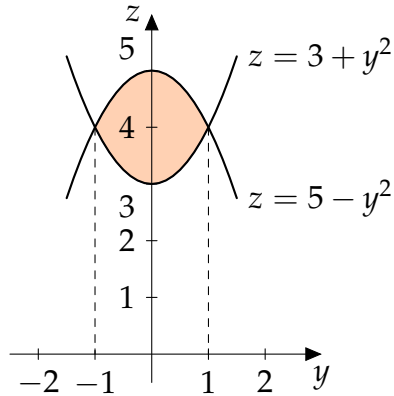
$$V = \iiint_{\Omega} dx dy dz.$$

**Example**

Compute the volume of the body bounded by cylindrical surfaces  $z = 5 - y^2$ ,  $z = y^2 + 3$  and planes  $x = 0$ ,  $x = 2$ .

While limits of variables  $x, z$  are obvious, we need to calculate intersections of cylindrical surfaces to determine the limits of variable  $y$ .

$$\begin{aligned} 5 - y^2 &= y^2 + 3 \\ 2y^2 &= 2 \\ y &= \pm 1. \end{aligned}$$



Therefore, inequalities for the domain  $\Omega$  are in the form

$$\Omega: \quad \begin{aligned} 0 &\leq x \leq 2, \\ -1 &\leq y \leq 1, \\ y^2 + 3 &\leq z \leq 5 - y^2. \end{aligned}$$

We calculate volume of the body by using the triple integral

$$\begin{aligned} V &= \iiint_{\Omega} dx dy dz = \int_0^2 dx \int_{-1}^1 dy \int_{y^2+3}^{5-y^2} dz = \int_0^2 dx \int_{-1}^1 [z]_{y^2+3}^{5-y^2} dy \\ &= \int_0^2 dx \cdot \int_{-1}^1 (2 - 2y^2) dy = 2 \left[ 2y - \frac{2y^3}{3} \right]_{-1}^1 = 2 \cdot \frac{8}{3} = \frac{16}{3}. \end{aligned}$$

## 87 – Practical applications of the triple integral, volume of a body

**Example**

Compute the volume of the body bounded by surfaces  $x^2 + y^2 + z^2 = r^2$  and  $x^2 - rx + y^2 = 0$ .

The equation  $x^2 + y^2 + z^2 = r^2$  describes spherical surface with the center in the coordinate origin and radius  $r$ . The second equation  $x^2 - rx + y^2 = 0$  can be rewritten to

$$\left(x - \frac{r}{2}\right)^2 + y^2 = \left(\frac{r}{2}\right)^2$$

which describes a cylindrical surface parallel to z-axis. Its projection to xy-plane is a ring with center  $S = \left[\frac{r}{2}, 0\right]$  and radius  $\frac{r}{2}$ . Intersection of both surfaces is well known Viviani's curve and you can see both surfaces and the domain on the figure.

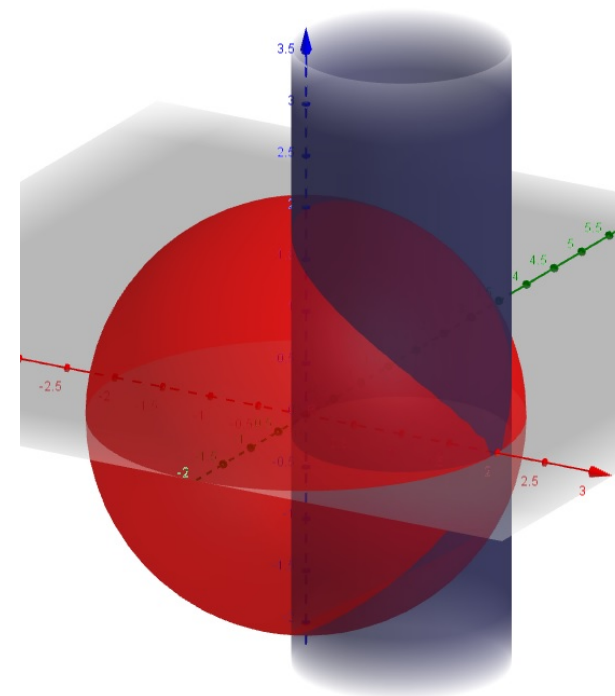
We will calculate the volume of the domain by transformation to cylindrical coordinates. The domain is symmetrical and therefore we will calculate only one quarter of the whole domain for  $y \geq 0, z \geq 0$ . The upper limit for parameter  $z$  is obtained by transformation of the equation of the sphere to cylindrical coordinates

$$z = \sqrt{r^2 - x^2 - y^2} = \sqrt{r^2 - \rho^2}.$$

The method of finding the limits for parameters  $\varphi, \rho$  with center in the point  $S$  was explained in the section Polar coordinates. By using such method we obtain for the transformation to cylindrical coordinates limits in the form

$$\begin{aligned} \Omega: \quad & 0 \leq \varphi \leq \frac{\pi}{2}, \\ & 0 \leq \rho \leq r \cos \varphi, \\ & 0 \leq z \leq \sqrt{r^2 - \rho^2}. \end{aligned}$$

We have to be careful with the order of integration. The limits of parameter  $z$  depends on radius  $\rho$ , while parameter  $\rho$  depends on azimuth  $\varphi$ . Therefore the first integration (inner integral) must be calculated with respect to  $z$  and the last one (outer integral) must be with respect to  $\varphi$ .



## 88 – Practical applications of the triple integral, volume of a body

Therefore the volume of the body from previous page is calculated by

$$V = 4 \iiint_{\Omega} dx dy dz = 4 \iiint_{\Omega} \rho d\varphi d\rho dz$$

$$= 4 \int_0^{\pi/2} d\varphi \int_0^{r \cos \varphi} d\rho \int_0^{\sqrt{r^2 - \rho^2}} \rho dz = 4 \int_0^{\pi/2} d\varphi \int_0^{r \cos \varphi} \rho \sqrt{r^2 - \rho^2} d\rho.$$

We solve this integral by substituting  $t = r^2 - \rho^2$  from which we obtain  $-\frac{1}{2} dt = \rho d\rho$ . We also transform the integration limits. For the lower limit  $\rho = 0$  it holds  $t = r^2$ , while for the upper limit  $\rho = r \cos \varphi$  similarly

$$t = r^2 - r^2 \cos^2 \varphi = r^2 \sin^2 \varphi.$$

The integration then follows

$$V = 4 \int_0^{\pi/2} d\varphi \int_{r^2}^{r^2 \sin^2 \varphi} -\frac{1}{2} \sqrt{t} dt = 2 \int_0^{\pi/2} \left[ \frac{2}{3} t^{3/2} \right]_{r^2 \sin^2 \varphi}^{r^2} d\varphi$$

$$= \frac{4}{3} \int_0^{\pi/2} (r^3 - r^3 \sin^3 \varphi) d\varphi = \frac{4}{3} r^3 \left( \int_0^{\pi/2} d\varphi - \int_0^{\pi/2} \sin^3 \varphi d\varphi \right).$$

We need another substitution for solving the second integral of trigonometrical function  $\sin^3 \varphi$ . By putting  $u = \cos \varphi$  and  $du = -\sin \varphi d\varphi$  and transforming of the integration limits  $u(0) = 1$ ,  $u\left(\frac{\pi}{2}\right) = 0$  we obtain

$$V = \frac{4}{3} r^3 \left( [\varphi]_0^{\pi/2} - \int_0^1 (1 - u^2) du \right)$$

$$= \frac{4}{3} r^3 \left( \frac{\pi}{2} - \left[ u - \frac{u^3}{3} \right]_0^1 \right) = \frac{2}{3} r^3 \left( \pi - \frac{4}{3} \right).$$



## 89 – Practical applications of the triple integral, volume of a body

## Example

Compute the volume of the body bounded by surfaces

$$x^2 + y^2 + (z - r)^2 = r^2 \text{ and } z = \sqrt{3x^2 + 3y^2}.$$

The equation  $x^2 + y^2 + (z - r)^2 = r^2$  describes the sphere with radius  $r$  and the center in point  $S = [0, 0, r]$ . The second equation  $z = \sqrt{3x^2 + 3y^2}$  describes the cone oriented along the  $z$ -axis with vertex in the coordinates origin. We will transform the problem to spherical coordinates. While the limits for azimuth  $\varphi$  must be  $0 \leq \varphi \leq 2\pi$ , we have to find also the limits for angle  $\vartheta$ . Let us make the projection of the domain to  $yz$ -plane, which is visible on the figure. By putting  $x = 0$  in the equation of the cone  $z = \sqrt{3x^2 + 3y^2}$  we obtain the equation of both lines

$$z = \sqrt{3y^2} = \sqrt{3}|y|.$$

Therefore, the upper limit for the angle  $\vartheta$  must fulfil

$$\tan \vartheta = \frac{y}{z} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Hence, the upper limit for  $\vartheta = \frac{\pi}{6}$ . We can also find the upper limit for radius  $\rho$ . We transform the equation of the sphere to  $x^2 + y^2 + z^2 - 2zr = 0$  and then to spherical coordinates

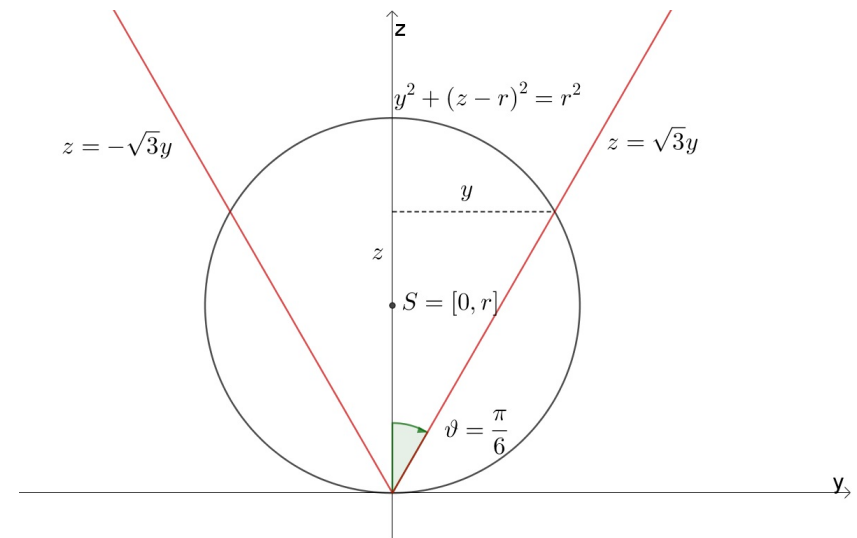
$$\begin{aligned} \rho^2 - 2r\rho \cos \vartheta &= 0 \\ \rho(\rho - 2r \cos \vartheta) &= 0 \end{aligned}$$

We have

$$\begin{aligned} \Omega^*: \quad 0 &\leq \varphi \leq 2\pi, \\ 0 &\leq \vartheta \leq \frac{\pi}{6}, \\ 0 &\leq \rho \leq 2r \cos \vartheta. \end{aligned}$$

The limits of variable  $\rho$  depends on variable  $\vartheta$ , therefore we have to start the calculation with inner integral with respect to variable  $\rho$ .

$$\begin{aligned} V &= \int_0^{2\pi} d\varphi \int_0^{\pi/6} d\vartheta \int_0^{2r \cos \vartheta} \rho^2 \sin \vartheta d\rho = 2\pi \int_0^{\pi/6} \sin \vartheta \left[ \frac{\rho^3}{3} \right]_0^{2r \cos \vartheta} d\vartheta \\ &= \frac{16}{3} \pi r^3 \int_0^{\pi/6} \sin \vartheta \cos^3 \vartheta d\vartheta = \left| \begin{array}{ll} t = \cos \vartheta & 0 \rightarrow 1 \\ dt = -\sin \vartheta d\vartheta & \frac{\pi}{6} \rightarrow \frac{\sqrt{3}}{2} \end{array} \right| \\ &= -\frac{16}{3} \pi r^3 \int_1^{\sqrt{3}/2} t^3 dt = \frac{16}{3} \pi r^3 \left[ \frac{t^4}{4} \right]_{\sqrt{3}/2}^1 = \frac{7}{12} \pi r^3. \end{aligned}$$



## 90 – Practical applications of the triple integral, volume of a body

### Exercise

Compute the volume of body bounded by surfaces  $z = x^2 + y^2$ ,  $z = y$ .

## 91 – Practical applications of the triple integral, volume of a body

### Exercise

Compute the volume of body bounded by surfaces  $x - y + z = 6$ ,  $x + y = 2$ ,  $x = y$ ,  $y = 0$ ,  $z = 0$ .

## 92 – Practical applications of the triple integral, volume of a body

### Exercise

Compute the volume of body bounded by surfaces  $y = x^2$ ,  $z = 0$ ,  $y + z = 2$ .

## 93 – Practical applications of the triple integral, volume of a body

### Exercise

Compute the volume of body bounded by surfaces  $y = \ln x$ ,  $y = \ln^2 x$ ,  $z = 0$ ,  $y + z = 1$ .

## 94 – Practical applications of the triple integral, mass of a body

**Mass of the body**  $\Omega$  is given by

$$m = \iiint_{\Omega} \sigma(x, y, z) \, dx \, dy \, dz$$

where  $\sigma(x, y, z) > 0$  denotes volume density in each point of the domain  $\Omega$ .

**Example**

Compute the mass of the body bounded by surfaces  $x^2 + y^2 + z^2 \leq 4$ . The volume density in each point of  $\Omega$  is equal to its distance to the coordinates origin.

The domain  $\Omega$  is the sphere with center in the coordinate origin and radius 2. Therefore we will calculate the problem by using transformation to spherical coordinates with transformation equations

$$\begin{aligned} x &= \rho \cos \varphi \sin \vartheta, \\ y &= \rho \sin \varphi \sin \vartheta, \\ z &= \rho \cos \vartheta. \end{aligned}$$

The volume density in each point of  $\Omega$  is equal to its distance to the coordinates origin, therefore

$$\sigma = \sqrt{x^2 + y^2 + z^2} = \rho.$$

After the transformation to spherical coordinates, the domain is rectangular given by inequalities

$$\begin{aligned} \Omega^* \quad 0 &\leq \rho \leq 2, \\ 0 &\leq \varphi \leq 2\pi, \\ 0 &\leq \vartheta \leq \pi. \end{aligned}$$

Now, we can calculate mass of the sphere

$$\begin{aligned} m &= \iiint_{\Omega} \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz = \iiint_{\Omega^*} \rho \cdot \rho^2 \sin \vartheta \, d\rho \, d\varphi \, d\vartheta \\ &= \int_0^{2\pi} d\varphi \cdot \int_0^{\pi} \sin \vartheta \, d\vartheta \cdot \int_0^2 \rho^3 \, d\rho = [\varphi]_0^{2\pi} \cdot [-\cos \vartheta]_0^{\pi} \cdot \left[ \frac{\rho^4}{4} \right]_0^2 \\ &= 2\pi \cdot 2 \cdot 4 = 16\pi. \end{aligned}$$

## 95 – Practical applications of the triple integral, mass of a body

### Exercise

Compute the mass of body  $x^2 + y^2 + z^2 \leq 1$  with density  $\sigma = \frac{2}{x^2 + y^2 + z^2}$ .

## 96 – Practical applications of the triple integral, mass of a body

### Exercise

Compute the mass of body bounded by surfaces  $x^2 = 2y$ ,  $y + z = 1$ ,  $2y + z = 2$ , with density  $\sigma = y$ .



## 97 – Practical applications of the triple integral, statical moments and moments of inertia

Let the domain  $\Omega$  is a body with given density  $\sigma(x, y, z) > 0$  in each point  $X = [x, y, z] \in \Omega$ .

**Statical moment of a body**  $S_{xy}$  or  $S_{xz}$  or  $S_{yz}$  to coordinate plane  $xy$  or  $xz$  or  $yz$  is defined by

$$S_{xy} = \iiint_{\Omega} z \sigma(x, y, z) dx dy dz,$$

$$S_{xz} = \iiint_{\Omega} y \sigma(x, y, z) dx dy dz,$$

$$S_{yz} = \iiint_{\Omega} x \sigma(x, y, z) dx dy dz.$$

The coordinates of **center of mass**  $C = [\xi, \eta, \zeta]$  can then be calculated by

$$\xi = \frac{S_{yz}}{m}, \quad \eta = \frac{S_{xz}}{m}, \quad \zeta = \frac{S_{xy}}{m},$$

where  $m$  is the mass of the body.

**Moment of inertia** of the body rotating around the  $x$ -axis resp.  $y$ -axis resp.  $z$ -axis is given by

$$I_x = \iiint_{\Omega} (y^2 + z^2) \sigma(x, y, z) dx dy dz,$$

$$I_y = \iiint_{\Omega} (x^2 + z^2) \sigma(x, y, z) dx dy dz,$$

$$I_z = \iiint_{\Omega} (x^2 + y^2) \sigma(x, y, z) dx dy dz.$$

## 98 – Practical applications of the triple integral, statical moments of a body

### Exercise

Calculate the statical moments of a body  $x^2 + y^2 + z^2 \leq 1$ ,  $x \geq 0$ ,  $y \geq 0$ ,  $z \geq 0$  to  $xy$ -plane. Consider constant density  $\sigma$ .

## 99 – Practical applications of the triple integral, statical moments of a body

### Exercise

Calculate the statical moments of a cone with radius of the base  $r = 3$  and height  $h = 2$  to plane that is parallel to the base going through the vertex of the cone. Consider constant density  $\sigma$ .

## 100 – Practical applications of the triple integral, moments of inertia of a body

### Exercise

Calculate the moments of inertia of the body bounded by surfaces  $x + 2y + 3z = 1$ ,  $x = 0$ ,  $y = 0$ ,  $z = 0$  rotating around  $y$ -axis. Consider constant density  $\sigma$ .

## 101 – Practical applications of the triple integral, moments of inertia of a body

### Exercise

Calculate the moments of inertia of the body bounded by surfaces  $x^2 + y^2 = z^2$ ,  $z = 1$  rotating around  $z$ -axis. Consider constant density  $\sigma$ .

# **Worksheets for Mathematics III**

Theory of the field

## 103 – Vector function

**Definition**

Let  $D \subseteq \mathbb{R}$ . A **vector function** of one real variable  $t \in D$  is defined as a function of one real variable whose range is a vector

$$\mathbf{f}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} = (x(t), y(t), z(t))$$

**Components**  $x(t), y(t), z(t)$  are real functions of variable  $t$ .

From the geometrical point of view the vector function  $\mathbf{f}(t)$  describes the set of points in three-dimensional space with coordinates  $[x(t), y(t), z(t)]$ ,  $t \in D$ . It will create the **graph of the vector function**.

If  $x(t), y(t), z(t)$  are continuous for each  $t \in D = [a, b]$ , then continuous vector function  $\mathbf{f}(t)$  defines three-dimensional curve, whose parametrical equations are given by  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$ ,  $t \in [a, b]$ . From the physical point of view the vector function represent the trajectory of moving mass point.

We can define all key concepts of calculus also for vector functions - limits, continuity, derivatives, indefinite and definite integral. The calculation is made for each component separately. We can also use all concepts of vector algebra for vector functions - operations with vectors, inner and vector product.

**Example**

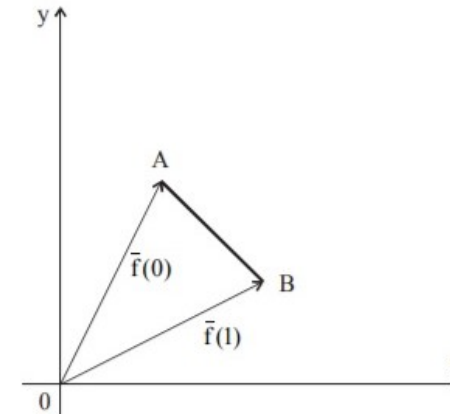
Draw the graph of the vector function

$$\mathbf{f} = (1 + t)\mathbf{i} + (2 - t)\mathbf{j}, \quad t \in [0, 1].$$

The function is continuous on its domain. The graph is a two-dimensional curve. Parametrical equations of curve

$$\begin{aligned} x &= 1 + t, \\ y &= 2 - t, \quad t \in [0, 1] \end{aligned}$$

describes the segment of line  $AB$ , given by  $A = [x(0), y(0)] = [1, 2]$  and  $B = [x(1), y(1)] = [2, 1]$ , see figure.



## 104 – Vector function

**Example**

Draw the graph of the vector function

$$\mathbf{f} = 3 \cos t \mathbf{i} + 3 \sin t \mathbf{j}, \quad t \in [0, 2\pi].$$

Function is continuous on its domain. The graph is a two-dimensional curve. Parametrical equations of curve

$$\begin{aligned} x &= 3 \cos t, \\ y &= 3 \sin t, \quad t \in [0, 2\pi] \end{aligned}$$

describes circle with center in the coordinate origin and radius  $r = 3$ , see figure. Starting and ending point of the curve is the same

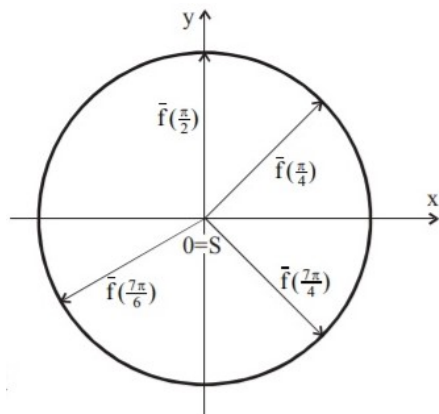
$$A = B = [x(0), y(0)] = [x(2\pi), y(2\pi)] = [3, 0].$$

We can prove it by raising both parametrical equations to the second power

$$x^2 = 9 \cos^2 t, \quad y^2 = 9 \sin^2 t$$

and summing them together

$$x^2 + y^2 = 9(\cos^2 t + \sin^2 t) = 9.$$

**Example**

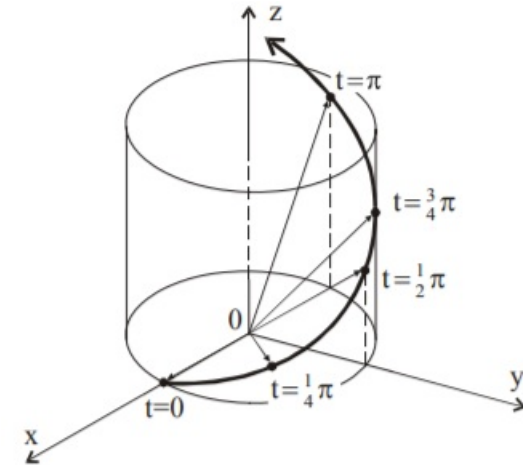
Draw the graph of the vector function

$$\mathbf{f} = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k}, \quad t \in [0, +\infty).$$

Function is continuous on its domain. The graph is a three-dimensional curve. Parametrical equations of curve

$$\begin{aligned} x &= \cos t, \\ y &= \sin t, \\ z &= t, \quad t \in [0, \infty) \end{aligned}$$

define the screw line with starting point  $[1,0,0]$  on cylindrical surface  $x^2 + y^2 = 1$ . Analogically to the previous example, we can obtain this equation by raising first two parametrical equations to the second power and summing them together, see figure.





## 105 – Vector function

### Exercise

Draw the graph of the vector function

a)  $\mathbf{f} = 2 \cos t \mathbf{i} + 3 \sin t \mathbf{j}, \quad t \in [0, 2\pi),$

b)  $\mathbf{f} = t^2 \mathbf{i} + t \mathbf{j}, \quad t \in (-\infty, +\infty).$

## 106 – Scalar field

**Definition**

**Scalar field** on the domain  $\Omega \subset \mathbb{R}^3$  is given by an scalar function  $u = u(x, y, z)$  defined on  $\Omega$ .

Scalar field assigns one real number (scalar) to each point in  $\Omega$ . The rate of change of the scalar field is given by directional derivative.

**Definition**

Let the scalar field  $u = u(x, y, z)$  is given on the domain  $\Omega$ , point  $A = [a_1, a_2, a_3] \in \Omega$  and unit vector  $\mathbf{s} = (s_1, s_2, s_3)$ . We define the limit

$$\lim_{t \rightarrow 0^+} \frac{u(A + t\mathbf{s}) - u(A)}{t}$$

as **directional derivative** of the scalar field  $u(x, y, z)$  in the point  $A$  along the vector  $\mathbf{s}$  and we denote it by  $\frac{du(A)}{ds}$ .

**Theorem**

Let partial derivatives of the scalar function  $u$  exist in the point  $A \in \Omega$ . The directional derivative of the scalar field  $u(x, y, z)$  in the point  $A$  along the unit vector  $\mathbf{s}$  can be written in the form

$$\frac{du(A)}{ds} = \frac{\partial u(A)}{\partial x} s_1 + \frac{\partial u(A)}{\partial y} s_2 + \frac{\partial u(A)}{\partial z} s_3.$$

The directional derivative of the scalar field  $u(x, y, z)$  in the point  $A$  along the vector  $\mathbf{s}$  determines the slope of the scalar field  $u(x, y, z)$  in the point  $A$  along the vector  $\mathbf{s}$ , i.e. rate of change of the scalar field  $u(x, y, z)$  in the point  $A$  in the direction of the vector  $\mathbf{s}$ .

## 107 – Scalar field

**Definition**

The vector function

$$\mathbf{grad} u = \frac{\partial u(x, y, z)}{\partial x} \mathbf{i} + \frac{\partial u(x, y, z)}{\partial y} \mathbf{j} + \frac{\partial u(x, y, z)}{\partial z} \mathbf{k} = \left( \frac{\partial u(x, y, z)}{\partial x}, \frac{\partial u(x, y, z)}{\partial y}, \frac{\partial u(x, y, z)}{\partial z} \right)$$

is called the **gradient of the scalar field**  $u(x, y, z)$ .

The direction of the greatest increase of the scalar field is given by gradient of the scalar field.

- By implementation of the **Hamilton operator (nabla operator)**

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

we can write the gradient of the scalar field  $u(x, y, z)$  in the form

$$\mathbf{grad} u = \nabla u.$$

- The directional derivative of the scalar field  $u(x, y, z)$  in the point  $A$  along the unit vector  $\mathbf{s}$  can be written in the form

$$\frac{du(A)}{ds} = \frac{\partial u(A)}{\partial x} s_1 + \frac{\partial u(A)}{\partial y} s_2 + \frac{\partial u(A)}{\partial z} s_3 = \mathbf{grad} u \cdot \mathbf{s}.$$

## 108 – Scalar field

**Theorem (Properties of the gradient)**

1. Gradient of the scalar field  $u(x, y, z)$  is perpendicular to the contours of the scalar field  $u(x, y, z)$  in each point  $A \in \Omega$ .
2. Gradient of the scalar field  $u(x, y, z)$  points in the direction of the greatest increase of the scalar field  $u(x, y, z)$ . The opposite direction is the greatest rate of decrease of the scalar field  $u(x, y, z)$ .
3. The value of the greatest increase of the scalar field  $u(x, y, z)$  is equal to  $|\mathbf{grad} u|$ .

**Example**

Find the directional derivative of the scalar field  $u = 3x^2 - 4y^3 + 2z^4$  in the point  $A = [1, 2, 1]$  along the vector  $\mathbf{s} = \mathbf{AB}$ , while  $B = [4, 6, 6]$ .

We need to find the values of the partial derivatives of the scalar field  $u(x, y, z)$  in the point  $A$ .

$$\frac{\partial u}{\partial x} = 6x, \quad \frac{\partial u}{\partial y} = -12y^2, \quad \frac{\partial u}{\partial z} = 8z^3,$$

$$\frac{\partial u(A)}{\partial x} = 6, \quad \frac{\partial u(A)}{\partial y} = -48, \quad \frac{\partial u(A)}{\partial z} = 8.$$

To find a unit vector  $\mathbf{s}$  in the direction of the  $\mathbf{AB}$  vector we need to calculate

$$\mathbf{AB} = B - A = (3, 4, 5),$$

$$|\mathbf{AB}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2},$$

$$\mathbf{s} = \frac{\mathbf{AB}}{|\mathbf{AB}|} = \left( \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}} \right) = \left( \frac{3\sqrt{2}}{10}, \frac{2\sqrt{2}}{5}, \frac{\sqrt{2}}{2} \right).$$

By using formula for the directional derivative we obtain

$$\frac{du(A)}{ds} = 6 \cdot \frac{3\sqrt{2}}{10} - 48 \cdot \frac{2\sqrt{2}}{5} + 8 \cdot \frac{\sqrt{2}}{2} = -\frac{67}{5}\sqrt{2}.$$

## 109 – Scalar field

**Example**

Find the gradient of the scalar field  $u = x^2 + y^2 + z^2 - 2xy + 2xz + 2yz$ , the unit direction  $\mathbf{s}$  of the greatest rate of increase of the field in the point  $A = [1, 2, 1]$  and the greatest value of directional derivative of the scalar field  $u$  in the point  $A$ .

We calculate the partial derivatives of the scalar field  $u(x, y, z)$ .

$$\frac{\partial u}{\partial x} = 2x - 2y + 2z, \quad \frac{\partial u}{\partial y} = 2y - 2x + 2z, \quad \frac{\partial u}{\partial z} = 2z + 2x + 2y.$$

Therefore the gradient vector is in the form

$$\mathbf{grad} u = (2x - 2y + 2z, 2y - 2x + 2z, 2z + 2x + 2y).$$

Because the gradient always points to the direction of the greatest increase of the scalar field  $u$ , we can calculate the unit vector  $\mathbf{s}$  by

$$\mathbf{grad} u(A) = (0, 4, 8),$$

$$|\mathbf{grad} u(A)| = \sqrt{80} = 4\sqrt{5},$$

$$\mathbf{s} = \frac{\mathbf{grad} u(A)}{|\mathbf{grad} u(A)|} = \left(0, \frac{4}{4\sqrt{5}}, \frac{8}{4\sqrt{5}}\right) = \left(0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right).$$

Now we are able to obtain the directional derivative  $\frac{du(A)}{ds}$  by

$$\begin{aligned} \frac{du(A)}{ds} &= \mathbf{grad} u(A) \cdot \mathbf{s} = (0, 4, 8) \cdot \left(0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right) \\ &= 0 + \frac{4\sqrt{5}}{5} + \frac{16\sqrt{5}}{5} = 4\sqrt{5}. \end{aligned}$$

We can compare the results and confirm that for the direction of the greatest increase of the scalar field  $u$  it holds

$$\frac{du(A)}{ds} = |\mathbf{grad} u(A)|.$$

## 110 – Scalar field

## Exercise

Calculate directional derivative of scalar field  $u$  in the point  $A$  along the unit vector  $\mathbf{s}$ :

a)  $u = 5x^4 - 4xy + 2y - 7$ ,  $A = [1, 1]$ ,  $\mathbf{s} = -\mathbf{i}$ ,

b)  $u = \sqrt{x^2 + y^2}$ ,  $A = [3, 4]$ ,  $\mathbf{s} \parallel \mathbf{v} = (4, -3)$ .

## 111 – Scalar field

### Exercise

- a) Find the points where gradient of the scalar field  $u = x^2 + 2xy + 4y^2 + z^2 - 4z$  is equal to zero.
- b) Find the direction of the greatest rate of increase of the scalar field  $u = x^2 + y^2 + z^2 - 1$  in the point  $A = [0, -2, 1]$ .

## 112 – Vector field

We often use vector fields to describe different physical phenomena. Vector field assigns to each point  $X = [x, y, z]$  in the domain  $\Omega$  the only vector  $\mathbf{f}$ , whose components are real functions  $P(x, y, z)$ ,  $Q(x, y, z)$ ,  $R(x, y, z)$ .

**Definition**

**Vector field** on the domain  $\Omega \subset \mathbb{R}^3$  is given by a vector function

$$\mathbf{f}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}.$$

**Definition**

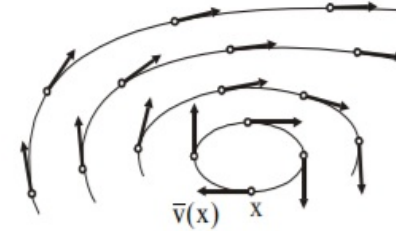
Let functions  $P(x, y, z)$ ,  $Q(x, y, z)$ ,  $R(x, y, z)$  be continuous on the domain  $\Omega$  and  $X = [x, y, z] \in \Omega$  be an arbitrary point. The vector field  $\mathbf{f} = (P(X), Q(X), R(X))$  is said to be **conservative** if and only if there exist a scalar field  $\Phi$  on  $\Omega$  such that

$$\mathbf{f} = \left( \frac{\partial \Phi(X)}{\partial x}, \frac{\partial \Phi(X)}{\partial y}, \frac{\partial \Phi(X)}{\partial z} \right) = \mathbf{grad} \Phi(x, y, z).$$

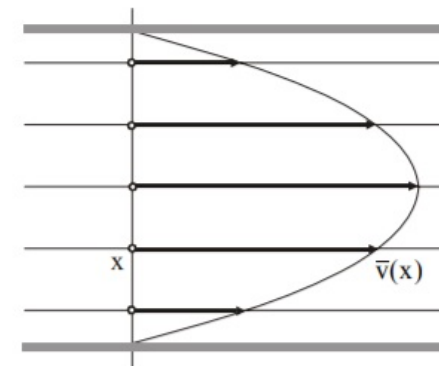
The scalar field  $\Phi$  is called a **scalar potential** of a vector function  $\mathbf{f}$ .

To describe a vector field we use lines of force, flow lines, etc. The vector field  $\mathbf{f}(X)$  always points in the direction of the tangent of such lines in each point  $X \in \Omega$ . See figures where you can find

- peripheral velocity of the rotational movement of the solid body



- velocity of laminar flow of the fluid





## 113 – Vector field

**Definition**

Let vector field be given by vector function

$$\mathbf{f}(X) = P(X)\mathbf{i} + Q(X)\mathbf{j} + R(X)\mathbf{k},$$

while functions  $P(X), Q(X), R(X)$  are continuous and have partial derivatives on  $\Omega$ .

- **Divergence of the vector field  $\mathbf{f}(X)$**  is defined as a scalar field

$$\operatorname{div} \mathbf{f}(X) = \nabla \cdot \mathbf{f}(X) = \frac{\partial P(X)}{\partial x} + \frac{\partial Q(X)}{\partial y} + \frac{\partial R(X)}{\partial z}.$$

- Vector field, where for all  $X \in \Omega$  holds  $\operatorname{div} \mathbf{f}(X) = 0$  is called **solenoidal** (divergence-free).
- Points  $X \in \Omega$ , where  $\operatorname{div} \mathbf{f}(X) > 0$  are called **sources**.
- Points  $X \in \Omega$ , where  $\operatorname{div} \mathbf{f}(X) < 0$  are called **sinks**.
- **Curl of the vector field  $\mathbf{f}(X)$**  is defined as a vector field

$$\begin{aligned} \operatorname{curl} \mathbf{f}(X) = \nabla \times \mathbf{f}(X) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(X) & Q(X) & R(X) \end{vmatrix} \\ &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right). \end{aligned}$$

- Vector field, where for all  $X \in \Omega$  holds  $\operatorname{curl} \mathbf{f}(X) = \mathbf{0}$  is called **irrotational** (curl-free).

We can clarify the meaning of the divergence and the curl of vector field on the vector field of velocity  $\mathbf{v}(x, y, z)$  of stationary flow of the fluid. Divergence of the vector field  $\mathbf{v}$  in point  $A$  describes the volume of the fluid that flows out from unit of volume in unit of time in the neighbourhood of point  $A$ , i.e. intensity of the source of unit volume. Curl of the vector field  $\mathbf{v}$  in point  $A$  defines the direction of the axis around which the fluid rotates in the neighbourhood of point  $A$ .

**Theorem**

Vector field  $\mathbf{f}(X) = P(X)\mathbf{i} + Q(X)\mathbf{j} + R(X)\mathbf{k}$  is conservative on  $\Omega$  if and only if it is irrotational on  $\Omega$ , i.e.  $\operatorname{curl} \mathbf{f}(X) = \mathbf{0}$ .

## 114 – Vector field

**Example**

Represent the vector field  $\mathbf{f}(x, y) = (x - y)\mathbf{i} + (x + y)\mathbf{j}$  given on  $\Omega: x^2 + y^2 \leq 4$ .

To represent the vector field we can choose some points in the domain  $\Omega$  and calculate appropriate values of the vector field  $\mathbf{f}$ .

$$A = [1, 1]: \quad \mathbf{f}(A) = 0 \cdot \mathbf{i} + 2 \cdot \mathbf{j} = (0, 2)$$

$$B = [2, 0]: \quad \mathbf{f}(B) = 2 \cdot \mathbf{i} + 2 \cdot \mathbf{j} = (2, 2)$$

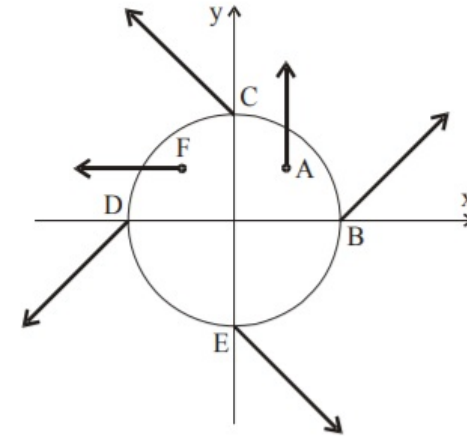
$$C = [0, 2]: \quad \mathbf{f}(C) = -2 \cdot \mathbf{i} + 2 \cdot \mathbf{j} = (-2, 2)$$

$$D = [-2, 0]: \quad \mathbf{f}(D) = -2 \cdot \mathbf{i} - 2 \cdot \mathbf{j} = (-2, -2)$$

$$E = [0, -2]: \quad \mathbf{f}(E) = 2 \cdot \mathbf{i} - 2 \cdot \mathbf{j} = (2, -2)$$

$$F = [-1, 1]: \quad \mathbf{f}(F) = -2 \cdot \mathbf{i} + 0 \cdot \mathbf{j} = (-2, 0)$$

The representation of the vector field is visible on following figure.



**Example**

Find out if the vector field  $\mathbf{f}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$  is

- solenoidal,
- irrotational.
- Find its scalar potential  $\Phi$  if exists.

For the vector field  $\mathbf{f}$  it holds

$$P = x^2, \quad Q = y^2, \quad R = z^2,$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 2y, \quad \frac{\partial P}{\partial z} = 2z.$$

- According to the definition of divergence we obtain

$$\operatorname{div} \mathbf{f}(x, y, z) = 2x + 2y + 2z,$$

therefore the vector field is not solenoidal. Points, where  $2x + 2y + 2z > 0$  are sources, while there are sinks in points where  $2x + 2y + 2z < 0$ .

For example the point  $A = [1, 1, 1]$  is source because  $\operatorname{div} \mathbf{f}(A) = 6 > 0$ . The point  $B = [-1, -1, -1]$  is sink because  $\operatorname{div} \mathbf{f}(B) = -6 < 0$ .

- Based on the definition of curl we calculate

$$\operatorname{curl} \mathbf{f}(X) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \left( \frac{\partial (z^2)}{\partial y} - \frac{\partial (y^2)}{\partial z}, \frac{\partial (x^2)}{\partial z} - \frac{\partial (z^2)}{\partial x}, \frac{\partial (y^2)}{\partial x} - \frac{\partial (x^2)}{\partial y} \right) = \mathbf{o}.$$

Therefore, the given field is irrotational and it is also conservative. That's why we are able to find its scalar potential.

- To find the scalar potential  $\Phi$  we use its properties from the definition of the scalar potential

$$P = \frac{\partial \Phi}{\partial x}, \quad Q = \frac{\partial \Phi}{\partial y}, \quad R = \frac{\partial \Phi}{\partial z}.$$

Hence

$$\Phi = \int P \, dx = \int x^2 \, dx = \frac{x^3}{3} + K_1(y, z),$$

where  $K_1(y, z)$  is an arbitrary function depending on variables  $y, z$ . To find it we derivate the scalar potential  $\Phi$  with respect to  $y$  and realise that

$$\frac{\partial K_1(y, z)}{\partial y} = y^2$$

from which we obtain

$$K_1(y, z) = \int y^2 \, dy = \frac{y^3}{3} + K_2(z),$$

where  $K_2(z)$  is an arbitrary function depending only on variable  $z$ , which must fulfil  $K_2'(z) = z^2$  based on the partial derivative of potential  $\Phi$  with respect to  $z$ . Therefore

$$K_2(z) = \int z^2 \, dz = \frac{z^3}{3} + C,$$

where  $C$  is an arbitrary real constant. Finally, we obtained scalar potential in the form

$$\Phi(X) = \frac{x^2 + y^2 + z^2}{3} + C.$$

## 117 – Vector field

**Example**

Vector field of the force  $\mathbf{F}$  in each point  $X = [x, y, z]$  points to the coordinates origin and its magnitude is equal to  $|\mathbf{F}| = \frac{1}{\rho^2}$ , where  $\rho$  is the distance between the point and the coordinate origin. Find out if the field is conservative.

The force  $\mathbf{F}$  has the same direction as the position vector  $\mathbf{r}$  of an arbitrary point  $X$ ,

$$\mathbf{r} = \mathbf{OX} = X - O = (x, y, z)$$

but the opposite orientation. Therefore

$$\mathbf{F} = (-cx, -cy, -cz),$$

where  $c > 0$  is an arbitrary constant. The distance of the point  $X = [x, y, z]$  is equal to

$$\rho = |\mathbf{OX}| = \sqrt{x^2 + y^2 + z^2}.$$

The magnitude of the force is given and is equal to

$$|\mathbf{F}| = \frac{1}{\rho^2} = \frac{1}{x^2 + y^2 + z^2}$$

from which we obtain

$$c = \frac{1}{\sqrt{(x^2 + y^2 + z^2)^3}}.$$

We have found the components of the force vector

$$\mathbf{F} = -\frac{x}{\sqrt{(x^2 + y^2 + z^2)^3}}\mathbf{i} - \frac{y}{\sqrt{(x^2 + y^2 + z^2)^3}}\mathbf{j} - \frac{z}{\sqrt{(x^2 + y^2 + z^2)^3}}\mathbf{k}.$$

According to the definition of the curl we calculate

$$\begin{aligned} \mathbf{curl} \mathbf{F} &= \left( \frac{3yz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} - \frac{3yz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} \right) \mathbf{i} \\ &+ \left( \frac{3xz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} - \frac{3xz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} \right) \mathbf{j} \\ &+ \left( \frac{3xy\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} - \frac{3xy\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} \right) \mathbf{k} = \mathbf{o}. \end{aligned}$$

The vector field is irrotational and therefore it is also conservative.

## 118 – Vector field

## Exercise

Find the divergence and the curl of the vector field  $\mathbf{f}$ :

a)  $\mathbf{f}(x, y, z) = x^2yz \mathbf{i} + xy^2z \mathbf{j} + xyz^2 \mathbf{k},$

b)  $\mathbf{f}(x, y, z) = \mathbf{grad} (x^3 + y^3 + z^3).$

## 119 – Vector field

## Exercise

Find out if following vector field  $\mathbf{f}$  is solenoidal, irrotational and conservative, find its scalar potential  $\Phi$  if exists:

a)  $\mathbf{f}(x, y, z) = (y + z) \mathbf{i} + (x + z) \mathbf{j} + (x + y) \mathbf{k}$ ,

b)  $\mathbf{f}(x, y, z) = \mathbf{grad}(xyz)$ .

# **Worksheets for Mathematics III**

Line integral



## 121 – The curve and its orientation

**Definition**

Let  $x = x(t)$ ,  $y = y(t)$ ,  $z = z(t)$  be continuous functions for  $t \in [a, b]$ . The **curve**  $k$  with parametrical equations

$$\begin{aligned}x &= x(t), \\y &= y(t), \\z &= z(t), \quad t \in [a, b]\end{aligned}$$

is called **positively oriented with respect to the parameter**  $t$ , if and only if its points are ordered so that for arbitrary values  $t_1, t_2 \in [a, b]$ ,  $t_1 < t_2$ , the point  $M_1 = [x(t_1), y(t_1), z(t_1)]$  lies before the point  $M_2 = [x(t_2), y(t_2), z(t_2)]$ , i.e.

$$\forall t_1, t_2 \in [a, b] : t_1 < t_2 \Leftrightarrow M_1 \prec M_2.$$

Reversely,

$$\forall t_1, t_2 \in [a, b] : t_1 < t_2 \Leftrightarrow M_2 \prec M_1,$$

the curve is called **negatively oriented with respect to the parameter**  $t$ .

**Remark**

The symbol  $\prec$  means "precedes" or "lies before".

**Definition**

If the curve  $k$  is positively oriented with respect to the parameter  $t \in [a, b]$ , then the point  $A = [x(a), y(a), z(a)]$  is called the **starting point** of the curve and the point  $B = [x(b), y(b), z(b)]$  is called the **ending point** of the curve.

## 122 – The curve and its orientation

**Definition**

Let curve  $k$  is given by parametrical equations

$$\begin{aligned}x &= x(t), \\y &= y(t), \\z &= z(t), \quad t \in [a, b]\end{aligned}$$

with starting point  $A = [x(a), y(a), z(a)]$  and ending point  $B = [x(b), y(b), z(b)]$ .

- The curve is called **closed**, if  $A \equiv B$ .
- The curve is called **smooth** on  $[a, b]$ , if there exists continuous derivatives of parametrical equations

$$\begin{aligned}\dot{x} &= \dot{x}(t), \\ \dot{y} &= \dot{y}(t), \\ \dot{z} &= \dot{z}(t)\end{aligned}$$

and  $\forall t \in [a, b] : (\dot{x}(t), \dot{y}(t), \dot{z}(t)) \neq (0, 0, 0)$ .

- The curve is called **piecewise smooth** on  $[a, b]$ , if it is smooth on  $[a, b]$  except for a finite number of points  $t_i \in [a, b], i = 1, \dots, n$ .
- The curve is called **simple** on  $[a, b]$ , if it doesn't intersect itself, i.e.

$$\forall t_1, t_2 \in (a, b) : t_1 \neq t_2 \Rightarrow M_1 \neq M_2.$$

## 123 – The curve and its orientation

**Example**

Write parametrization of the line segment  $\overline{AB}$ , where  $A = [0,0]$ ,  $B = [1,1]$ .

There are infinitely many possibilities how to write down a parametrization. For example:

1. If we consider the given segment as a part of graph of function  $y = x$ , we can put  $t = x = y$  and obtain

$$\begin{aligned}x &= t, \\y &= t, \quad t \in [0,1].\end{aligned}$$

The curve is positively oriented. For  $t = 0$  we obtain  $A = [x(0), y(0)] = [0,0]$  and analogically for  $t = 1$  we obtain  $B = [x(1), y(1)] = [1,1]$ .

2. It is not necessary to keep  $x = t$ . We can use parametrization

$$\begin{aligned}x &= r - 1, \\y &= r - 1, \quad r \in [1,2].\end{aligned}$$

which is also positively oriented. We obtain  $A = [x(1), y(1)] = [0,0]$ , while  $B = [x(2), y(2)] = [1,1]$ .

3. If we use following parametrization

$$\begin{aligned}x &= -s, \\y &= -s, \quad s \in [-1,0].\end{aligned}$$

The curve is then negatively oriented. In such situation  $B = [x(-1), y(-1)] = [1,1]$ , while  $A = [x(0), y(0)] = [0,0]$ .

## 124 – Parametrization of the line segment

Parametrizations of the line segment between points  $A = [a_1, a_2, a_3]$ ,  $B = [b_1, b_2, b_3]$  are in the form

$$\begin{aligned}x &= a_1 + u_1 \cdot t, \\y &= a_2 + u_2 \cdot t, \\z &= a_3 + u_3 \cdot t, \quad t \in [0, 1].\end{aligned}$$

where  $\mathbf{u} = \mathbf{AB} = (u_1, u_2, u_3)$  is the vector parallel to the line segment  $AB$ .

**Example**

Write parametrization of the line segment between points  $A = [1, 2, \pi]$  and  $B = [8, -12, 0]$ .

We compute vector

$$\mathbf{AB} = B - A = (7, -14, -\pi).$$

The parametrical equations are

$$\begin{aligned}x &= 1 + 7t, \\y &= 2 - 14t, \\z &= \pi - \pi t, \quad t \in [0, 1].\end{aligned}$$

## 125 – Parametrization of the circle

Parametrical equations of the circle

$$(x - m)^2 + (y - n)^2 = r^2, \quad r > 0,$$

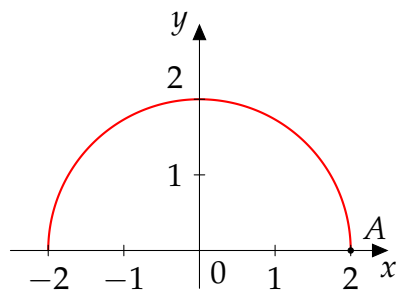
with the center in the point  $C = [m, n]$  and radius  $r$  are in the form

$$x = m + r \cos t,$$

$$y = n + r \sin t, \quad t \in [0, 2\pi].$$

**Example**

Compute parametrization of the circle with the center in the origin and radius  $r = 2$  for  $y \geq 0$ . The starting point of the curve is  $A = [2, 0]$ .



The parametrical equations are

$$x = 2 \cos t,$$

$$y = 2 \sin t, \quad t \in [0, \pi].$$

The curve is positively oriented with respect to the parameter  $t$ .

We can also express variable  $y$  from equation

$$x^2 + y^2 = 4$$

and obtain

$$y = \pm \sqrt{4 - x^2}.$$

For  $y > 0$  we consider only

$$y = \sqrt{4 - x^2}.$$

By putting  $s = x$  we then obtain parametrical equations of the given curve in the form

$$x = s,$$

$$y = \sqrt{4 - s^2}, \quad s \in [-2, 2].$$

The curve is negatively oriented with respect to parameter  $s$ .

## 126 – Parametrization of the ellipse

Parametrical equations of the ellipse

$$\frac{(x - m)^2}{a^2} + \frac{(y - n)^2}{b^2} = 1, \quad a, b > 0,$$

with the center in the point  $[m, n]$  and semi-axis  $a, b$  are in the form

$$x = m + a \cos t,$$

$$y = n + b \sin t, \quad t \in [0, 2\pi].$$

**Example**

Compute parametrization of the curve  $9x^2 + 4y^2 + 18x - 32y + 37 = 0$ .

We can find the center of the ellipse and sizes of the semi-axes by following calculation

$$9x^2 + 4y^2 + 18x - 32y + 37 = 0$$

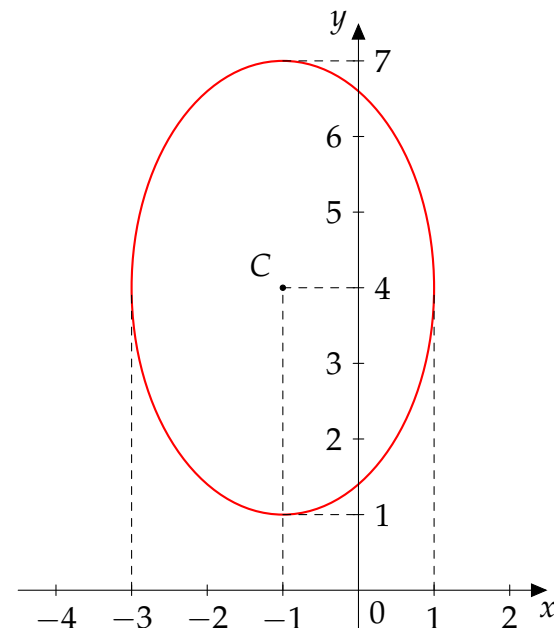
$$9(x^2 + 2x) + 4(y^2 - 8y) = -37$$

$$9(x^2 + 2x + 1) - 9 + 4(y^2 - 8y + 16) - 64 = -37$$

$$9(x + 1)^2 + 4(y - 4)^2 = 36$$

$$\frac{(x + 1)^2}{4} + \frac{(y - 4)^2}{9} = 1$$

The center of the ellipse is in point  $C = [-1, 4]$  and semi-axis are  $a = 2$ ,  $b = 3$



and the parametrical equations are

$$x = -1 + 2 \cos t,$$

$$y = 4 + 3 \sin t, \quad t \in [0, 2\pi].$$

## 127 – Line integral of a scalar field

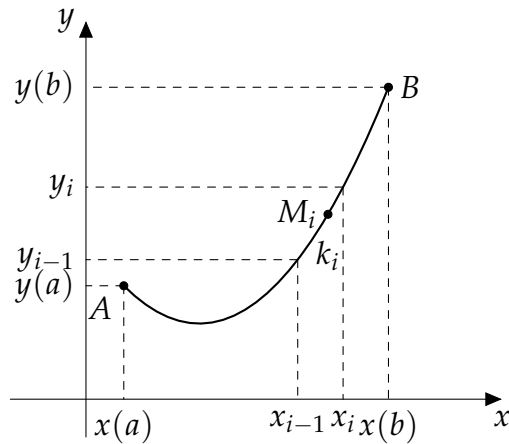
We need to divide our domain (the curve  $k$ ) into small elements. Let us consider the simple smooth curve  $k$  with parametrization

$$\begin{aligned}x &= x(t), \\y &= y(t), \\z &= z(t), \quad t \in [a, b]\end{aligned}$$

positively oriented with respect to the parameter  $t$ . We divide interval  $[a, b]$  by sequence of points

$$a = t_0 < t_1 < \dots < t_n = b$$

into  $n$  partial curves  $k_1, k_2, \dots, k_n$  according to the figure



For each  $i = 1, \dots, n$  we denote by  $\Delta s_i$  the length of each element  $k_i$  and we choose an arbitrary point  $M_i = [x(t_i), y(t_i), z(t_i)]$  in each element  $k_i$ . The curve  $k$  lies within a domain  $\Omega$  and we consider a bounded continuous scalar function  $u(X) = u(x, y, z)$  defined for each  $X \in \Omega$ . Now we can create the sum of products

$$\sum_{i=1}^n u(M_i) \cdot \Delta s_i = \sum_{i=1}^n u(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i$$

and define the line integral of a scalar field.

**Definition**

If there exists

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n u(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i$$

for  $n \rightarrow \infty$  and  $\Delta s_i \rightarrow 0$  we call it the **line integral of a scalar field  $u(x, y, z)$  along the curve  $k$**  and denote it

$$\int_k u(x, y, z) \, ds.$$

## 128 – Properties of the line integral of a scalar field

**Theorem (Properties of the line integral of a scalar field)**

$$1. \int_k c u(X) \, ds = c \int_k u(X) \, ds,$$

$$2. \int_k (u(X) + v(X)) \, ds = \int_k u(X) \, ds + \int_k v(X) \, ds,$$

$$3. \int_k u(X) \, ds = \int_{k_1} u(X) \, ds + \int_{k_2} u(X) \, ds,$$

where  $c \in \mathbb{R}$ ,  $k_1, k_2$  are non-overlapping curves such that curve  $k$  fulfils  $k = k_1 \cup k_2$  and  $u(X), v(X)$  are bounded continuous scalar functions for all  $X \in \Omega$  that contains the curve  $k$ .



## 129 – Line integral of a scalar field

**Remark**

The line integral of the scalar field doesn't depend on the orientation of the curve, because the lengths of all components  $\Delta s_i$  are always positive.

The element of the curve  $ds$  in the three-dimensional space is the body diagonal of the rectangular hexahedron with sides  $dx$ ,  $dy$ ,  $dz$ . Therefore we obtain

$$\begin{aligned} ds &= \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{(\dot{x}(t) dt)^2 + (\dot{y}(t) dt)^2 + (\dot{z}(t) dt)^2} \\ &= \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2} dt. \end{aligned}$$

The line integral of the scalar field can then be written in the form

$$\int_k u(x, y, z) ds = \int_a^b u(x(t), y(t), z(t)) \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2} dt.$$

That way we transform the line integral of the scalar field into the one-dimensional definite integral.

**Example**

Calculate the line integral  $\int_k (x + z) ds$  along the line segment between points  $A = [1, 2, 3]$ ,  $B = [3, 2, 1]$ .

We create the parametrization of the line segment

$$\begin{aligned} x &= 1 + 2t, \\ y &= 2, \\ z &= 3 - 2t, \quad t \in [0, 1] \end{aligned}$$

and calculate its derivatives

$$\dot{x} = 2, \quad \dot{y} = 0, \quad \dot{z} = -2.$$

We express the element  $ds$

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2} dt = \sqrt{8} dt = 2\sqrt{2} dt$$

to calculate the line integral

$$\int_k (x + z) ds = \int_0^1 (1 + 2t + 3 - 2t) \cdot 2\sqrt{2} dt = 8\sqrt{2} \int_0^1 dt = 8\sqrt{2}.$$

## 130 – Line integral of a scalar field

If we consider just the two-dimensional problem, i.e. the curve is in  $xy$ -plane, the element

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} dt$$

and the line integral of the scalar field is then in the form

$$\int_k u(x, y) ds = \int_a^b u(x(t), y(t)) \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} dt.$$

**Example**

Calculate the line integral  $\int_k y^2 ds$ , where  $k$  is a circle with the center in the origin of coordinates and radius 4.

The parametrical equations of the circle are

$$\begin{aligned} x &= 2 \cos t, \\ y &= 2 \sin t, \quad t \in [0, 2\pi]. \end{aligned}$$

We calculate derivatives

$$\begin{aligned} \dot{x} &= -2 \sin t, \\ \dot{y} &= 2 \cos t \end{aligned}$$

and the element of the curve

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} dt = \sqrt{4 \sin^2 t + 4 \cos^2 t} dt = \sqrt{4} dt = 2 dt.$$

We are able to calculate the integral

$$\begin{aligned} \int_k y^2 ds &= \int_0^{2\pi} 4 \sin^2 t \cdot 2 dt = 8 \int_0^{2\pi} \frac{1}{2} (1 - \cos(2t)) dt \\ &= 4 \left[ t - \frac{1}{2} \sin(2t) \right]_0^{2\pi} = 8\pi. \end{aligned}$$

## 131 – Line integral of a scalar field

**Example**

Calculate the line integral  $\int_k y \, ds$ , where  $k$  is a part of the function  $y = x^3$  between points  $A = [0, 0]$ ,  $B = [1, 1]$ .

Parametrical equations of function  $y = x^3$ ,  $x \in [0, 1]$  are

$$\begin{aligned}x &= t, \\y &= t^3, \quad t \in [0, 1].\end{aligned}$$

We need to express a derivative of parametrical equations

$$\begin{aligned}x &= t, \\y &= 3t^2\end{aligned}$$

an element of the curve

$$ds = \sqrt{1 + (3t^2)^2} \, dt = \sqrt{1 + 9t^4} \, dt$$

to calculate the integral

$$\int_k y \, ds = \int_0^1 t^3 \sqrt{1 + 9t^4} \, dt.$$

In such integral we can use substitution

$$\begin{aligned}1 + 9t^4 &= z \\36t^3 \, dt &= dz\end{aligned}$$

to obtain

$$\int_k y \, ds = \frac{1}{36} \int_1^{10} \sqrt{z} \, dz = \frac{1}{36} \cdot \frac{2}{3} \left[ \sqrt{z^3} \right]_1^{10} = \frac{1}{54} (10\sqrt{10} - 1).$$

## 132 – Line integral of a scalar field

## Exercise

Compute the line integrals of scalar fields along given curves:

a)  $\int_k \frac{z^2}{x^2 + y^2} ds$ ,  $k$  is one thread of the spiral  $x = \cos t$ ,  $y = \sin t$ ,  $z = t$ ,  $t \in [0, 2\pi]$ ,

b)  $\int_k x ds$ ,  $k$  is a line segment between points  $A = [0, 0]$ ,  $B = [1, 2]$ .

## 133 – Line integral of a scalar field

## Exercise

Compute the line integrals of scalar fields along given curves:

a)  $\int_k x^2 ds$ ,  $k$  is an upper half of the circle  $x^2 + y^2 = a^2$ ,  $a > 0$ ,

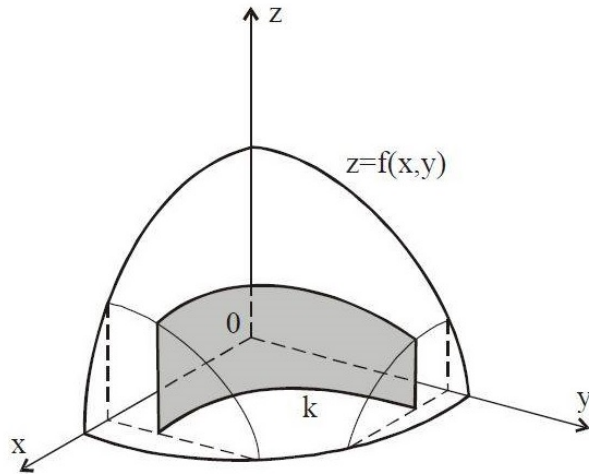
b)  $\int_k x^2 ds$ ,  $k : y = \ln x$ ,  $x \in [1, 3]$ .

## 134 – Practical applications of line integral, area of a cylindrical region

Let function  $f(x, y) \geq 0$  be continuous on a domain  $\Omega$  that contains the curve  $k$ . We consider the cylindrical surface between the plane  $z = 0$  and  $z = f(x, y)$  above the curve  $k$ , see figure. The area of such a surface is

$$A = \int_k f(x, y) ds.$$

This is the geometrical meaning of the line integral of a scalar field.

**Example**

Calculate the area of the cylindrical surface  $x^2 + y^2 = r^2$  bounded by  $z \geq 0$  and  $z \leq x$ .

The curve  $k$  is a part of the circle  $x^2 + y^2 = r^2$  for  $x \geq 0$ . Therefore, the parametrization of the curve  $k$  is in the form

$$\begin{aligned} x &= r \cos t, \\ y &= r \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

We express derivatives of the parametric equations

$$\begin{aligned} \dot{x} &= -r \sin t, \\ \dot{y} &= r \cos t \end{aligned}$$

and the element of the curve is then given by

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} dt = \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} dt = r dt.$$

Now we can calculate the area of given cylindrical surface

$$A = \int_k x ds = \int_{-\pi/2}^{\pi/2} r \cos t \cdot r dt = r^2 [\sin t]_{-\pi/2}^{\pi/2} = 2r^2.$$

## 135 – Practical applications of line integral, area of a cylindrical region

**Exercise**

Calculate the area of cylindrical surfaces bounded by given conditions:

a)  $x^2 + y^2 = r^2, z \geq 0, z \leq \frac{xy}{2r}, x \geq 0, y \geq 0,$

b)  $9y^2 = 4(x - 1)^3, z \geq 0, z \leq 2 - \sqrt{x}.$

## 136 – Practical applications of line integral, area of a cylindrical region

**Exercise**

Calculate the area of cylindrical surfaces bounded given conditions:

a)  $y^2 = 2x, z \geq 0, z \leq \sqrt{2x - 4x^2},$

b)  $y = \frac{3}{8}x^2, z \geq 0, z \leq x, x \geq 0, y \leq 6.$



## 137 – Practical applications of line integral, length of a curve

Let  $k$  be simple, piecewise smooth curve. The length of the curve is numerically equal to the area of the cylindrical surface above the curve  $k$  that is bounded by planes  $z = 0$  and  $z = 1$ . Hence, by letting  $f(x, y) = 1$  in formula for area of a cylindrical region  $A = \int_k f(x, y) ds$  we obtain

$$L = \int_k ds.$$

**Remark**

The length of the curve  $k$  in three dimensional space can be calculated using the same formula

$$L = \int_k ds.$$

**Example**

Calculate the length of one period of the cycloid  $x = a(t - \sin t)$ ,  
 $y = a(1 - \cos t)$ ,  $t \in [0, 2\pi]$ .

We need to calculate the derivatives of the parametric equations of the cycloid

$$\begin{aligned}\dot{x} &= a(1 - \cos t), \\ \dot{y} &= a \sin t.\end{aligned}$$

Further, we use them to express the element of the curve

$$\begin{aligned}ds &= \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} dt = \sqrt{a^2(1 - \cos t)^2 + a^2 \sin^2 t} dt \\ &= a \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt = a \sqrt{2 - 2 \cos t} dt = \sqrt{2}a \sqrt{1 - \cos t} dt.\end{aligned}$$

Now, we need to use trigonometric identity  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$  and obtain

$$ds = \sqrt{2}a \sqrt{2 \sin^2 \frac{t}{2}} dt = 2a \sin \frac{t}{2} dt.$$

Finally, we are able to calculate the length of the curve

$$L = \int_k ds = \int_0^{2\pi} 2a \sin \frac{t}{2} dt = 2a \left[ -2 \cos \frac{t}{2} \right]_0^{2\pi} = -4a \cdot (-1 - 1) = 8a.$$

## 138 – Practical applications of line integral, length of a curve

## Exercise

Calculate the lengths of the given curves:

a) cardioid with parametrical equations  $x = 2a \cos t - a \cos 2t$ ,  $y = 2a \sin t - a \sin 2t$ ,  $t \in [0, 2\pi]$ ,  $a > 0$ ,

b)  $y = \frac{1}{2} \ln x$ ,  $z = \frac{1}{2} x^2$ ,  $x \in [1, 2]$ .

## 139 – Practical applications of line integral, length of a curve

## Exercise

Calculate the lengths of the curves:

a)  $y = 1 - \ln \cos x, x \in \left[0, \frac{\pi}{4}\right],$

b)  $y = \frac{1}{2}x^2, z = \frac{1}{6}x^3, x \in [0, 1].$

## 140 – Practical applications of line integral, mass of a curve

Let  $k$  be a simple, piecewise smooth curve and continuous function  $\rho(x, y, z) > 0$  be its linear density. The mass of a curve (e.g. mass of a wire) is given by the line integral of a scalar field

$$m = \int_k \rho(x, y, z) ds.$$

**Example**

Calculate the mass of one thread of the screw line  $k : x = \cos t, y = \sin t, z = t, t \in [0, 2\pi]$  if its density is given by  $\rho = x^2 + y^2 + z^2$ .

Derivatives of the parametric equations are in the form

$$\begin{aligned}\dot{x} &= -\sin t, \\ \dot{y} &= \cos t, \\ \dot{z} &= 1.\end{aligned}$$

We express element of the curve

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2} dt = \sqrt{\sin^2 t + \cos^2 t + 1^2} dt = \sqrt{2} dt.$$

Now we can use the line integral and calculate the mass of the given curve

$$\begin{aligned}m &= \int_k (x^2 + y^2 + z^2) ds = \sqrt{2} \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2) dt \\ &= \sqrt{2} \int_0^{2\pi} (t^2 + 1) dt = \sqrt{2} \left[ \frac{t^3}{3} + t \right]_0^{2\pi} \\ &= \sqrt{2} \left( \frac{8\pi^3}{3} + 2\pi \right) = 2\sqrt{2}\pi \left( \frac{4}{3}\pi^3 + 1 \right).\end{aligned}$$

## 141 – Practical applications of line integral, mass of a curve

## Exercise

- a) Calculate the mass of one quarter of the circle  $x = a \sin t$ ,  $y = a \cos t$ ,  $t \in \left[0, \frac{\pi}{2}\right]$  if the density in each point is equal to its  $y$ -coordinate.
- b) Calculate the mass of the parabola  $y = \frac{1}{2}x^2$  between the points  $A = \left[1, \frac{1}{2}\right]$  and  $B = [2, 2]$ . The density  $\rho = \frac{y}{x}$ .

## 142 – Practical applications of line integral, mass of a curve

### Exercise

- a) Calculate the mass of the curve  $y = \ln x$ , where  $x \in [1, 2]$ . The density at each point is equal to the square of its  $x$ -coordinate.
- b) Calculate the mass of the catenary  $y = \frac{a}{2} \left( e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right)$  for  $x \in [0, a]$ ,  $a > 0$ . The density  $\rho = \frac{a}{y}$ .

## 143 – Line integral of a vector field

Let us consider the simple smooth curve  $k$  with parametrization

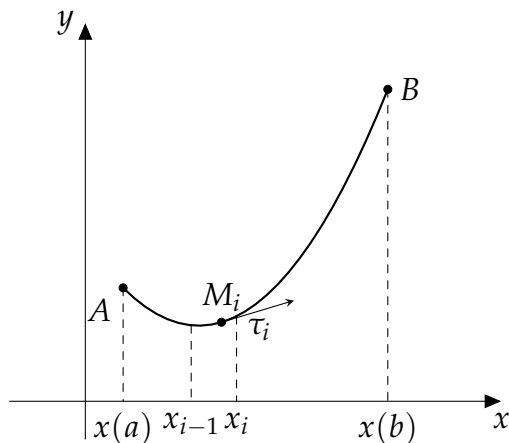
$$\begin{aligned}x &= x(t), \\y &= y(t), \\z &= z(t), \quad t \in [a, b]\end{aligned}$$

that is positively oriented with respect to the parameter  $t$ . The curve lies within the domain  $\Omega$ .

We divide interval  $[a, b]$  by sequence of points

$$a = t_0 < t_1 < \dots < t_n = b$$

into  $n$  partial curves  $k_1, k_2, \dots, k_n$ . We are also able to construct positively oriented unitary tangential vector  $\tau_i(M_i)$  at each point  $M_i$  according to the figure.



Furthermore, we consider bounded continuous vector field

$$\mathbf{F}(X) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

defined for each  $X \in \Omega$ . We create scalar products of the vector field and the positively oriented tangential vector of each element of the curve  $\mathbf{F}(M_i) \cdot \Delta\mathbf{s}_i$ , where  $\Delta\mathbf{s}_i = \Delta s_i \tau_i$ . Now we can create sum of such products

$$\sum_{i=1}^n \mathbf{F}(M_i) \cdot \Delta\mathbf{s}_i = \sum_{i=1}^n \mathbf{F}(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i \tau_i$$

and define the line integral of a vector field.

**Definition**

If there exists

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbf{F}(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i \tau_i$$

for  $n \rightarrow \infty$  and  $\Delta s_i \rightarrow 0$ , we call it the **line integral of a vector field  $\mathbf{F}(x, y, z)$  along the curve  $k_+$**  and denote it

$$\int_{k_+} \mathbf{F}(x, y, z) \cdot d\mathbf{s},$$

where  $k_+$  denotes curve  $k$  positively oriented with respect to parameter  $t$ . While  $k_-$  would denote curve  $k$  negatively oriented with respect to parameter  $t$ .

## 144 – Line integral of a vector field

**Remark**

The line integral of the vector field depends on the orientation of the curve  $k$ , because coordinates of the unitary tangential vectors  $\tau_i$  depend on the orientation of the curve.

To derive the form of the line integral of the vector field, we need to express the inner product

$$\begin{aligned}\mathbf{F}(x, y, z) \cdot d\mathbf{s} &= (P(x, y, z), Q(x, y, z), R(x, y, z)) \cdot (dx, dy, dz) \\ &= P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz\end{aligned}$$

and the differentials  $dx = \dot{x}(t) dt$ ,  $dy = \dot{y}(t) dt$ ,  $dz = \dot{z}(t) dt$  by using parametrization of the curve.

The line integral of the vector field then can be written in the form

$$\begin{aligned}\int_k \mathbf{F}(x, y, z) \cdot d\mathbf{s} &= \varepsilon \int_a^b [P(x(t), y(t), z(t))\dot{x}(t) + Q(x(t), y(t), z(t))\dot{y}(t) \\ &\quad + R(x(t), y(t), z(t))\dot{z}(t)] dt,\end{aligned}$$

where  $\varepsilon = 1$  in case of positively oriented curve  $k$  with respect to the parameter  $t$ , while  $\varepsilon = -1$  in case of negatively oriented curve  $k$ .

This way we transform the line integral of the vector field into the one-dimensional definite integral, similarly to the case of the line integral of a scalar field.

**Theorem (Properties of the line integral of a vector field)**

1.  $\int_k c \mathbf{F}(X) \cdot d\mathbf{s} = c \int_k \mathbf{F}(X) \cdot d\mathbf{s},$
2.  $\int_k (\mathbf{F}(X) + \mathbf{G}(X)) \cdot d\mathbf{s} = \int_k \mathbf{F}(X) \cdot d\mathbf{s} + \int_k \mathbf{G}(X) \cdot d\mathbf{s},$
3.  $\int_k \mathbf{F}(X) \cdot d\mathbf{s} = \int_{k_1} \mathbf{F}(X) \cdot d\mathbf{s} + \int_{k_2} \mathbf{F}(X) \cdot d\mathbf{s},$
4.  $\int_{k_+} \mathbf{F}(X) \cdot d\mathbf{s} = - \int_{k_-} \mathbf{F}(X) \cdot d\mathbf{s}.$

where  $c \in \mathbb{R}$ ,  $k_1, k_2$  are non-overlapping curves such that curve  $k$  fulfils  $k = k_1 \cup k_2$  (considering the same orientation of these curves) and  $\mathbf{F}(X), \mathbf{G}(X)$  are bounded continuous vector functions for all  $X \in \Omega$  that contains the curve  $k$ .



## 145 – Line integral of a vector field

**Example**

Calculate the line integral of the vector field  $\mathbf{F} = (x, y, z)$  along the curve  $k$ , that is one thread of the spiral  $x = 2 \cos t$ ,  $y = 2 \sin t$ ,  $z = 3t$ ,  $t \in [0, 2\pi]$ . The curve is positively oriented with respect to the parameter  $t$ .

We express derivatives of parametrization equations

$$\begin{aligned}\dot{x} &= -2 \sin t \\ \dot{y} &= 2 \cos t \\ \dot{z} &= 3\end{aligned}$$

and we can calculate the integral

$$\begin{aligned}\int_k \mathbf{F}(x, y, z) \cdot d\mathbf{s} &= \int_k x dx + y dy + z dz \\ &= \int_0^{2\pi} [2 \cos t \cdot (-2 \sin t) + 2 \sin t \cdot 2 \cos t + 3t \cdot 3] dt \\ &= \int_0^{2\pi} [-4 \sin t \cos t + 4 \sin t \cos t + 9t] dt = \int_0^{2\pi} 9t dt = \frac{9}{2} [t^2]_0^{2\pi} = 18\pi^2.\end{aligned}$$

## 146 – Line integral of a vector field

If we consider the two-dimensional problem, i.e. the curve is in  $xy$ -plane, the vector field  $\mathbf{F} = (P(x, y), Q(x, y))$  and the tangential vector of the element

$$d\mathbf{s} = (dx, dy) = (\dot{x}(t) dt, \dot{y}(t) dt),$$

the line integral of the vector field is then in the form

$$\int_k \mathbf{F}(x, y) \cdot d\mathbf{s} = \varepsilon \int_a^b [P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t)] dt.$$

**Example**

Calculate the line integral  $\int_k (x + y) dx + (x - y) dy$ , where

$k : y = \frac{1}{x}, x \in [2, 3]$ . The starting point of the curve  $k$  is  $A = \left[2, \frac{1}{2}\right]$ .

Parametrical equations of the curve  $k$  are

$$\begin{aligned} x &= t, \\ y &= \frac{1}{t}, \quad t \in [2, 3]. \end{aligned}$$

The curve is positively oriented with respect to the parameter  $t$ . We calculate the derivatives

$$\begin{aligned} \dot{x} &= 1, \\ \dot{y} &= -\frac{1}{t^2} \end{aligned}$$

and the integral

$$\begin{aligned} \int_k (x + y) dx + (x - y) dy &= \int_2^3 \left[ \left(t + \frac{1}{t}\right) + \left(t - \frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right) \right] dt \\ &= \int_2^3 \left[ t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^3} \right] dt = \int_2^3 \left( t + \frac{1}{t^3} \right) dt = \left[ \frac{t^2}{2} - \frac{1}{2t^2} \right]_2^3 \\ &= \frac{9}{2} - \frac{1}{18} - 2 + \frac{1}{8} = \frac{185}{72}. \end{aligned}$$

## 147 – Line integral of a vector field

## Exercise

Compute the line integrals of vector fields along given curves:

a)  $\int_k x \, dx - y \, dy + z \, dz$ ,  $k$  is an oriented line segment  $\overline{AB}$ :  $A = [1, 1, 1]$ ,  $B = [4, 3, 2]$ ,

b)  $\int_k y \, dx + x \, dy$ ,  $k$  is one quarter of the circle  $x = a \cos t$ ,  $y = a \sin t$ ,  $a > 0$ ,  $t \in \left[0, \frac{\pi}{2}\right]$  with starting point  $A [a, 0]$ .

## 148 – Line integral of a vector field

## Exercise

Compute the line integrals of vector fields along given curves:

a)  $\int_k (xy - 1) dx + x^2y dy$ ,  $k$  is an arc of the ellipse  $x = \cos t$ ,  $y = 2 \sin t$  from the starting point  $A = [1, 0]$  to the end point  $B = [0, 2]$ ,

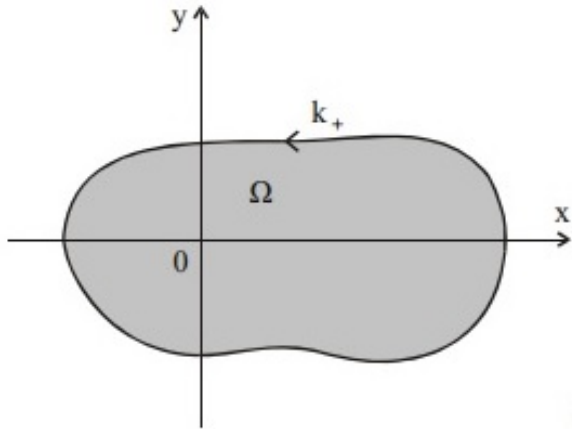
b)  $\int_k xy dx + (y - x) dy$ ,  $k$  is a part of the parabola  $y^2 = x$  from the starting point  $A = [0, 0]$  to the end point  $B = [1, 1]$ .

## 149 – Green's theorem

Green's theorem expresses the relation between a line integral of a vector field along a plane (two-dimensional) closed curve and a double integral.

**Definition**

Let  $\Omega$  be a domain in a plane bounded by a simple closed curve  $k$ . The curve  $k$  is **orientated positively** if traveling on the curve we always have got the domain  $\Omega$  on the left side, see figure.

**Remark**

Positive orientation of the closed curve means traveling in a counter-clockwise direction, while negative orientation means traveling in a clockwise direction.

We denote the line integral of a vector field  $\mathbf{F}(X)$  along a closed curve  $k$  by

$$\oint_k \mathbf{F}(X) \cdot d\mathbf{s}.$$

**Theorem (Green's theorem)**

Let two-dimensional vector field

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

have continuous partial derivatives on the plane domain  $\Omega$ , which is bounded by simple piecewise smooth closed positively orientated curve  $k$ . Then

$$\oint_k P(x, y) dx + Q(x, y) dy = \iint_{\Omega} \left[ \frac{\partial Q(x, y)}{\partial x} - \frac{\partial P(x, y)}{\partial y} \right] dx dy.$$

The Green's theorem transforms the line integral of a vector field along a plane closed curve to a double integral over the domain  $\Omega$ , that is bounded by the curve  $k$ . It is especially useful in situations when  $k$  is a polygon and we would have to calculate as many line integrals as the lines the polygon consists of.

## 150 – Green's theorem

## Example

Calculate the integral  $\oint_k (2xy - 5y) dx + (x^2 + y) dy$ , where  $k$  is the positively orientated circle with the center in the origin of coordinates and radius  $r$ .

The curve  $k$  is simple and closed. Also both functions  $P(x, y) = 2xy - 5y$  and  $Q(x, y) = x^2 + y$  fulfil assumptions of the Green's theorem. We calculate derivatives

$$\begin{aligned}\frac{\partial P(x, y)}{\partial y} &= 2x - 5 \\ \frac{\partial Q(x, y)}{\partial x} &= 2x\end{aligned}$$

and evaluate the integral by using Green's theorem.

$$\oint_k (2xy - 5y) dx + (x^2 + y) dy = \iint_{\Omega} (2x - (2x - 5)) dx dy = \iint_{\Omega} 5 dx dy = 5\pi r^2.$$

## 151 – Green's theorem

**Example**

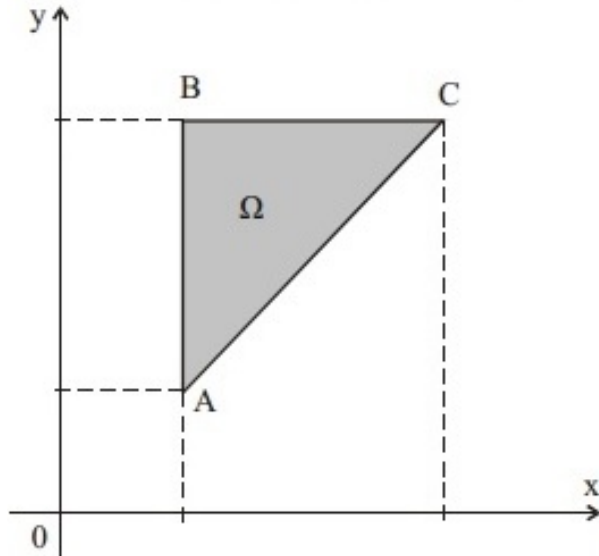
Calculate the integral  $\oint_k (x^2 + y^2) dx + (x + y)^2 dy$ , where  $k$  consists of the sides of the triangle ABC:  $A = [1, 1], B = [1, 3], C = [3, 3]$ . The curve is positively orientated.

All assumptions of the Green's theorem are fulfilled.

$$P = x^2 + y^2, \quad Q = (x + y)^2,$$

$$\frac{\partial P(x, y)}{\partial y} = 2y, \quad \frac{\partial Q(x, y)}{\partial x} = 2(x + y).$$

The domain is shown on following figure.



We calculate the double integral as a normal one with respect to the  $x$ -axis with inequalities for  $\Omega$  in the form

$$\Omega: \quad 1 \leq x \leq 3,$$

$$x \leq y \leq 3.$$

By using Green's theorem we obtain

$$\oint_k (x^2 + y^2) dx + (x + y)^2 dy = \iint_{\Omega} (2x + 2y - 2y) dx dy = 2 \int_1^3 dx \int_x^3 x dy$$

$$= 2 \int_1^3 x [y]_x^3 dx = 2 \int_1^3 (3x - x^2) dx = 2 \left[ \frac{3}{2}x^2 - \frac{x^3}{3} \right]_1^3 = \frac{20}{3}.$$

## 152 – Green's theorem

### Exercise

Calculate integral  $\oint_k (x^2 + y^2) dy$  along sides of the rectangle  $0 \leq x \leq 2, 0 \leq y \leq 4$  by using the Green's theorem. The curve is positively orientated.



## 153 – Green's theorem

## Exercise

Calculate integral  $\oint_k 2y \, dx - (x + y) \, dy$  by using the Green's theorem. The curve  $k$  consists of the sides of the triangle  $x \geq 0, y \geq 0, x + 2y \leq 4$ . The curve is positively orientated.

## 154 – Green's theorem

### Exercise

Calculate integral  $\oint_k (x + y) dx - (x - y) dy$  along the positively orientated ellipse  $4x^2 + 9y^2 = 36$  by using the Green's theorem.

## 155 – Green's theorem

### Exercise

Calculate integral  $\oint_k (e^x \sin y - 16y) dx + (e^x \cos y + 16) dy$ , where  $k$  is positively orientated circle  $x^2 + y^2 = 2x$  by using the Green's theorem.

## 156 – Path independence of line integral

We can use another method of calculation of the line integral of a vector field in situations when the value of the integral doesn't depend on an integration curve and depends only on its starting and ending point.

**Definition**

Let points  $A, B \in \Omega$ . Let the vector function  $\mathbf{F}(X)$  be continuous over the domain  $\Omega$ . If the value of the line integral of the vector field

$$\int_k \mathbf{F}(X) \cdot ds$$

doesn't depend on the integration curve  $k$  with starting point  $A$  and ending point  $B$  that lies within the domain  $\Omega$ , we say **the integral is path independent between points  $A, B$** .

If this property is fulfilled for arbitrary points  $A, B \in \Omega$ , we say **integral is path independent over  $\Omega$** .

## 157 – Path independence of line integral

**Theorem (Path independence of line integral)**

Let  $\mathbf{F}(X) = P(X)\mathbf{i} + Q(X)\mathbf{j} + R(X)\mathbf{k}$  have continuous partial derivatives in a domain  $\Omega$ . Let a curve  $k$  lie within  $\Omega$ ,  $A$  be the starting point of the curve  $k$ , while  $B$  be its ending point. Then:

1. The line integral  $\int_k \mathbf{F}(X) \cdot d\mathbf{s} = \int_k P(X) dx + Q(X) dy + R(X) dz$  is path independent over  $\Omega$  if and only if there exists some scalar function  $\Phi(x, y, z)$  over  $\Omega$  such that  $\mathbf{F}(x, y, z) = \mathbf{grad} \Phi(x, y, z)$ , i.e.

$$P = \frac{\partial \Phi}{\partial x}, \quad Q = \frac{\partial \Phi}{\partial y}, \quad R = \frac{\partial \Phi}{\partial z}.$$

The vector field  $\mathbf{F}(x, y, z)$  is a **conservative field**, function  $\Phi(x, y, z)$  is a **scalar potential**.

2. The line integral  $\int_k \mathbf{F}(X) \cdot d\mathbf{s} = \int_k P(X) dx + Q(X) dy + R(X) dz$  is path independent over  $\Omega$  if and only if  $\mathbf{curl} \mathbf{F}(X) = \mathbf{o}$ , i.e.

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

3. In such case, the line integral of the vector field  $\mathbf{F}(X)$  along a curve  $k$  from the starting point  $A$  to the ending point  $B$  is given by the difference of the values of scalar potential in the end point  $B$  and starting point  $A$  :

$$\int_k \mathbf{F}(X) \cdot d\mathbf{s} = \int_k P(X) dx + Q(X) dy + R(X) dz = \Phi(B) - \Phi(A).$$

4. Hence, if the line integral is path independent and the curve  $k$  is closed then

$$\oint_k \mathbf{F}(X) \cdot d\mathbf{s} = 0.$$

## 158 – Path independence of line integral

If the problem is considered only in  $xy$ -plane, the path independent line integral of the vector field is given by

$$\int_k \mathbf{F}(X) \cdot d\mathbf{s} = \int_k P(x, y) dx + Q(x, y) dy = \Phi(B) - \Phi(A).$$

Two-dimensional vector field is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

We will show the way of calculation of the scalar potential  $\Phi(x, y)$  at following examples.

## 159 – Path independence of line integral

**Example**

Calculate the integral  $\int_k (3x^2 - 2xy + y^2) dx - (x^2 - 2xy + 3y^2) dy$ , where  $k$  is oriented line segment  $\overline{AB}$ ,  $A = [1, 2], B = [3, 1]$ .

The test condition  $\frac{\partial P}{\partial y} = -2x + 2y = \frac{\partial Q}{\partial x}$  is fulfilled.

To find the potential we first integrate  $P(x, y) = 3x^2 - 2xy + y^2$  with respect to  $x$ .

$$\Phi(x, y) = \int P(x, y) dx = \int (3x^2 - 2xy + y^2) dx = x^3 - x^2y + xy^2 + K(y),$$

where  $K(y)$  is a function of variable  $y$ . We determine it by setting the partial derivative  $\frac{\partial \Phi}{\partial y}$  equal to  $Q(x, y)$ . We have

$$\frac{\partial \Phi}{\partial y} = -x^2 + 2xy + K'(y) = -x^2 + 2xy - 3y^2 = Q(x, y).$$

Hence,  $K'(y) = -3y^2$  and

$$K(y) = - \int 3y^2 dy = -y^3 + C,$$

where  $C$  is a real constant. The scalar potential is in the form

$$\Phi(x, y) = x^3 - x^2y + xy^2 - y^3 + C.$$

We calculate the integral according to path independence theorem:

$$\Phi(B) = 3^3 - 3^2 \cdot 1 + 3 \cdot 1^2 - 1^3 = 20, \quad \Phi(A) = 1^3 - 1^2 \cdot 2 + 1 \cdot 2^2 - 2^3 = -5,$$

$$\int_k (3x^2 - 2xy + y^2) dx - (x^2 - 2xy + 3y^2) dy = \Phi(B) - \Phi(A) = 25.$$

## 160 – Path independence of line integral

**Example**

Calculate the integral  $\oint x^2 dx + y^2 dy$  along the positively orientated closed curve  $k: x^2 + y^2 = r^2$ .

Functions  $P = x^2$  and  $Q = y^2$  fulfil assumptions of path independence theorem. Partial derivatives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0.$$

Hence, the integral is path independent. The curve is closed. Therefore, the integral must be equal to zero.

$$\oint_k x^2 dx + y^2 dy = 0$$



## 161 – Path independence of line integral

## Example

Calculate the integral  $\int_k (2x + yz) dx + (xz + z^2) dy + (xy + 2yz) dz$  from the starting point  $A = [1, 1, 1]$  to the ending point  $B = [1, 2, 3]$ .

We determine functions  $P, Q, R$  and all needed partial derivatives:

$$P(x, y, z) = 2x + yz, \quad \frac{\partial P(x, y, z)}{\partial y} = z, \quad \frac{\partial P(x, y, z)}{\partial z} = y,$$

$$Q(x, y, z) = xz + z^2, \quad \frac{\partial Q(x, y, z)}{\partial x} = z, \quad \frac{\partial Q(x, y, z)}{\partial z} = x + 2z,$$

$$R(x, y, z) = xy + 2yz, \quad \frac{\partial R(x, y, z)}{\partial x} = y, \quad \frac{\partial R(x, y, z)}{\partial y} = x + 2z.$$

We can see that the test conditions are fulfilled

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \quad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

Hence,  $\text{curl } \mathbf{F} = \mathbf{0}$  and integral is path independent.

To find the scalar potential  $\Phi$  we use its properties from the definition of the scalar potential

$$P = \frac{\partial \Phi}{\partial x}, \quad Q = \frac{\partial \Phi}{\partial y}, \quad R = \frac{\partial \Phi}{\partial z}.$$

Hence,

$$\Phi = \int P dx = \int (2x + yz) dx = x^2 + xyz + K_1(y, z),$$

where  $K_1(y, z)$  is an arbitrary function depending on variables  $y, z$ . We determine it by setting the partial derivative  $\frac{\partial \Phi}{\partial y}$  equal to  $Q$ .

We obtain

$$\frac{\partial \Phi}{\partial y} = xz + \frac{\partial K_1(y, z)}{\partial y} = xz + z^2 = Q$$

and

$$K_1(y, z) = \int z^2 dy = yz^2 + K_2(z),$$

where  $K_2(z)$  is an arbitrary function depending only on variable  $z$ . We have

$$\Phi = x^2 + xyz + yz^2 + K_2(z).$$

We use equation  $\frac{\partial \Phi}{\partial z} = R$  and we obtain

$$xy + 2yz + K_2'(z) = xy + 2yz.$$

Integrating this equation we get

$$K_2(z) = \int 0 dz = C,$$

where  $C$  is an arbitrary real constant. Finally, we obtained scalar potential in the form

$$\Phi(X) = x^2 + xyz + yz^2 + C.$$

We calculate the integral according to path independence theorem:

$$\Phi(B) = 1^2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3^2 = 25, \quad \Phi(A) = 1^2 + 1 \cdot 1 \cdot 1 + 1 \cdot 1^2 = 3,$$

$$\int_k (2x + yz) dx + (xz + z^2) dy + (xy + 2yz) dz = \Phi(B) - \Phi(A) = 22.$$

## 162 – Path independence of line integral

## Exercise

Prove that following integrals are path independent. Then, calculate them if  $A$  is the starting point and  $B$  is the ending point.

a)  $\int_k \frac{x dx + y dy}{x^2 + y^2}, \quad A = [1, 2], B = [2, 3]$

b)  $\int_k (2y - 6xy^3) dx + (2x - 9x^2y^2) dy, \quad A = [1, 1], B = [4, 1]$

## 163 – Path independence of line integral

## Exercise

Prove that following integrals are path independent. Then, calculate them if  $A$  is the starting point and  $B$  is the ending point.

a)  $\int_k yz \, dx + xz \, dy + xy \, dz, \quad A = [2, 2, 2], B = [2, 3, 4]$

b)  $\int_k \frac{dx + 2dy + 3dz}{x + 2y + 3z}, \quad A = [0, 1, 0], B = [1, 0, 1]$

## 164 – Practical applications of line integral, work in a force field

Suppose an object moving in a force field  $\mathbf{F}$  along a curve  $k$ . Work done by the force  $\mathbf{F}$  is then given by the line integral of a vector field

$$W = \int_k \mathbf{F} \cdot d\mathbf{s}.$$

**Example**

Calculate the work done by the force field  $\mathbf{F} = (xy, x + y)$  on an object moving along the line segment  $\overline{AB}$  from the point  $A = [0, 0]$  to the point  $B = [1, 1]$ .

We describe the line segment by parametrization

$$\begin{aligned} x &= t, \\ y &= t, \quad t \in [0, 1] \end{aligned}$$

The curve is positively oriented with respect to the parameter  $t$ . We calculate the derivatives

$$\begin{aligned} \dot{x} &= 1, \\ \dot{y} &= 1 \end{aligned}$$

and calculate the work by using line integral of a vector field

$$\begin{aligned} W &= \int_k \mathbf{F} \cdot d\mathbf{s} = \int_k xy \, dx + (x + y) \, dy \\ &= \int_0^1 (t^2 + 2t) \, dt = \left[ \frac{t^3}{3} + t^2 \right]_0^1 = \frac{1}{3} + 1 = \frac{4}{3}. \end{aligned}$$

## 165 – Practical applications of line integral, work in a force field

### Exercise

- a) Calculate the work done by the force field  $\mathbf{F} = (xy, x + y)$  on an object moving along the curve  $k: x = y^2$  from the point  $A = [0, 0]$  to the point  $B = [1, 1]$ .
- b) Calculate the work done by the force field  $\mathbf{F} = (x + y, 2x)$  on an object moving along the closed curve  $k: x^2 + y^2 = r^2$  in a positive direction.

## 166 – Practical applications of line integral, work in a force field

## Exercise

- a) Calculate the work done by the force field  $\mathbf{F} = (x^2, y^2, z^2)$  on an object moving around the screw line  $k: x = \cos t, y = \sin t, z = t, t \in [0, \frac{\pi}{2}]$  in positive direction with respect to the parameter  $t$ .
- b) Calculate the work done by the force field  $\mathbf{F} = \mathbf{grad}(\Phi)$ ,  $\Phi = \ln x^2 + y^2 - \arctan \frac{x}{y}$  on an object moving from the point  $A = [1, 1]$  to the point  $B = [\sqrt{2}, \sqrt{2}]$ .

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