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Faculty of Mining and Geology

MATHEMATICS II

Pavel Kreml

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Department of Mathematics and Descriptive Geometry

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1. INDEFINITE INTEGRALS

We will turn our attention to reversing the operation of differentiation. Given the derivate of a function, we can find the function. This process is called **antidifferentiation**.

1.1. The Indefinite Integral

The derivative of function $f(x) = 5x^4$ at any value x is $f'(x) = 20x^3$. We put a question now. If $F'(x) = 5x^4$, what is $F(x)$? We know the function could be $F(x) = x^5$ because $F'(x)=(x^5)' = 5x^4 = f(x)$. But the function could also be $F(x) = x^5 + C$, because $F'(x) = (x^5 + C)' = 5x^4$. Thus we say the antiderivative of $f(x) = 5x^4$ is $F(x) = x^5 + C$, where C is an arbitrary constant.

The process of finding an antiderivative is called **integration**. The function that results when integration takes place is called an **indefinite integral** or more simply an **integral**. We can denote the indefinite integral (that is, the antiderivative) of function $f(x)$ by $\int f(x)dx$. Thus we can write $\int 5x^4 dx$ to indicate the antiderivative of the function $f(x) = 5x^4$. The expression is read as "the integral of $5x^4$ with respect to x ". In this case, $5x^4$ is called the integrand, the integral sign $\int$ indicates the process of integration, and the dx indicates that the integral is to be taken with respect to x. We can write $\int 5x^4 dx = x^5 + C$.

Definition: A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f. The **set** of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by $\int f(x)dx = F(x) + C$.

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

1.2. Computation of Integrals

Now that we know what are integrals. The next problem is to find out how to do them. For differentiation we had a tidy set of rules that allowed us to work out the derivative of just about any function that we cared to write down. The procedure is basically mechanical and can be done quite well by computers. There is nothing like this for integration. Integration is more of a skill than a routine.

If you can spot a function which differentiates to give your function then you have found an integral. Look at the following simple examples.

Example: What is the antiderivative of *f(x)* for $x \in (-\infty, +\infty)$:

a) $f(x) = 0$,
$F(x) = C$, because $F'(x) = C' = 0 = f(x)$.

b) $f(x) = 1$,
$F(x) = x + C$, because $F'(x) = (x + C)' = 1 + 0 = f(x)$.

c) $f(x) = x$,
$F(x) = \frac{x^2}{2} + C$, because $F'(x) = (\frac{x^2}{2} + C)' = x + 0 = x = f(x)$.

d) $f(x) = \sin x$,
$F(x) = -\cos x + C$, because $F'(x) = (-\cos x + C)' = -(-\sin x) + 0 = \sin x = f(x)$.

Table 1: Integration of some common functions

[1.] $\int 0 dx = C$

[2.] $\int 1 dx = x + C$

[3.] $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ where $x > 0$, $n \neq -1$

[4.] $\int \frac{1}{x} dx = \ln|x| + C$ where $x \neq 0$

[5.] $\int \sin x dx = -\cos x + C$

[6.] $\int \cos x dx = \sin x + C$

[7.] $\int \frac{1}{\cos^2 x} dx = \operatorname{tg} x + C$ where $x \neq (2k+1)\frac{\pi}{2}$, $k \in Z$

[8.] $\int \frac{1}{\sin^2 x} dx = -\operatorname{cotg} x + C$ where $x \neq k\pi$, $k \in Z$

[9.] $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$ where $x \in (-1, 1)$

[10.] $\int \frac{1}{1+x^2} dx = \operatorname{arctg} x + C$

[11.] $\int a^x dx = \frac{a^x}{\ln a} + C$ where $a > 0$, $a \neq 1$

[12.] $\int e^x dx = e^x + C$

This is a short table of some standard integrals. You can add in a constant of integration if you want to.

1.3. Some Properties of the Indefinite Integral

These are immediate consequences of the corresponding properties of derivatives. In each equation there is really an arbitrary constant of integration hanging around.

Let $f(x)$ and $g(x)$ be functions and k a constant. Then

- $\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$,
- $\int (f(x)-g(x))dx = \int f(x)dx - \int g(x)dx$,
- $\int kf(x)dx = k\int f(x)dx$, $k \in \mathrm{R}$

These rules may allow us to reduce an integral to the point where we can spot the answer.

Example:

$$\int\left(\cos x - \frac{2}{\sin^2 x} + 7e^x - 6\right)dx = \int \cos x dx - 2\int \frac{1}{\sin^2 x}dx + 7\int e^x dx - 6\int dx =$$

$$= \sin x - 2(-\cot gx) + 7e^x - 6x + C = \sin x + 2\operatorname{cotg} x + 7e^x - 6x + C .$$

1.4. Substitution

Many integrals are hard to perform at first hand. A smart idea consists in “cleaning”' them through an algebraic substitution which transforms the given integrals into easier ones.

Theorem: Let f be a continuous function defined on a interval (a,b) and let $\varphi:(a,b)\to(c,d)$ be a differentiable function. Then the following statement hold $\int f(x)dx = \int f(\varphi(t))\,\dot{\varphi}(t)dt$.

Remark: To express integration by substitution, we use the notation $x=\varphi(t)$*, then*

$$dx = \dot{\varphi}(t)dt \text{ and } \int f(x)dx = \int f(\varphi(t))\,\dot{\varphi}(t)dt.$$

Example: Find $\int \frac{\sqrt{1-x^2}}{x^2}dx$.

Solution: It is easy to see that sine-substitution is the one to use. Set $x=\varphi(t)=\sin t$. The function φ is continuous on $\left(0,\frac{\pi}{2}\right)$. Indeed, we have $dx = \dot{\varphi}(t)dt = \cos t\, dt$ and therefore $\int \frac{\sqrt{1-x^2}}{x^2}dx = \int \frac{\sqrt{1-\sin^2 t}}{\sin^2 t}\cos t dt = \int \frac{\cos^2 t}{\sin^2 t}dt = \int \left(\frac{1}{\sin^2 t}-1\right)dt = -\operatorname{cotg} t - t + C$

This will not answer fully the problem because the answer should be given as a function of x. Since $t = \arcsin x$ we get after easy simplifications

$$\int \frac{\sqrt{1-x^2}}{x^2}dx = -\operatorname{cotg} t - t + C = -\frac{\sqrt{1-\sin^2 t}}{\sin t} - t + C = -\frac{\sqrt{1-x^2}}{x} - \arcsin x + C \ .$$

Example: Find $\int x(x^2+7)^{55}dx$.

Solution: It is clear that once we develop the $(x^2+7)^{55}$ through the binomial formula, we will get a polynomial function easy to integrate. But it is clear that this will take a lot of time with big possibility of doing mistakes !

Let us consider the substitution $t = x^2 + 7$ (the reason behind is the presence of x in the integral since the derivative of (x^2+7) is $2x$). Indeed, we have $dt = 2x\,dx$ and therefore

$$\int x(x^2+7)^{55}dx = \int t^{55}\frac{dt}{2} = \frac{1}{112}t^{56} + C$$

Indefinite integral $\int x(x^2+7)^{55}dx$ is a function of x not of t. Therefore, we have to go back and replace t by $t(x)$:

$$\int x(x^2+7)^{55}dx = \frac{1}{112}t^{56} + C = \frac{1}{112}(x^2+7)^{56} + C$$

Example: Let us evaluate $\int e^{kx}dx$.

Solution: If you substitute $t = kx$, then $dt = (kx)'\,dx = kdx$, $dx = \frac{1}{k}dt$ and we obtain:

$$\int e^t\left(\frac{1}{k}dt\right) = \frac{1}{k}\int e^t dt = \frac{1}{k}e^t + C = \frac{1}{k}e^{kx} + C \ .$$

Similarly we can extend Table 1.

Table 1 (additional): Integration of some functions

[13.] $\int e^{kx}dx = \frac{1}{k}e^{kx} + C$

[14.] $\int \sin kx dx = -\frac{1}{k}\cos kx + C$

[15.] $\int \cos kx dx = \frac{1}{k}\sin kx + C$

[16.] $\int \frac{f'(x)dx}{f(x)} = \ln|f(x)| + C$

1.4. Integration by Parts

One of very common mistake students usually do is $\int f(x)g(x)dx = \int f(x)dx \int g(x)dx$

To convince yourself that it is a wrong formula, take $f(x) = x$ and $g(x)=1$. Therefore, one may wonder what to do in this case. A partial answer is given by what is called **Integration by Parts**.

In order to understand this technique, recall the formula

$(u(x)v(x))' = u'(x)v(x) + u(x)v'(x)$

which implies

$u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$

Theorem: Let *u(x)* and *v(x)* have continuous derivatives on (a,b).
Then $\int u'(x).v(x)\,dx = u(x).v(x) - \int u(x).v'(x)\,dx$

Remark: *We usually use the notation* $\int u'.vdx = u.v - \int u.v'dx$ *alternatively*

$\int u.v'dx = u.v - \int u'.vdx$.

Therefore if one of the two integrals $\int u(x).v'(x)\,dx$ and $\int u'(x).v(x)\,dx$ is easy to evaluate, we can use it to get the other one. This is the main idea behind integration by parts.

Example: Evaluate $\int x^2 e^x dx$.

Solution: Since the derivative or the integral of e^x lead to the same function, it will not matter whether we do one operation or the other. Therefore, we concentrate on the other function x^2. Clearly, if we integrate we will increase the power. This suggests that we should differentiate x^2 and integrate e^x. Hence

$u' = e^x, \quad v = x^2$, after integration and differentiation, we get

$u = \int u'dx = \int e^x dx = e^x, \; v' = (x^2)' = 2x$. The integration by parts formula gives

$\int x^2.e^x dx = e^x.x^2 - \int 2x.e^x dx$

It is clear that the new integral $\int 2x.e^x dx$ is not easily obtainable. Due to its similarity with the initial integral, we will use integration by parts for a second time. The same discussion as before leads to

$u' = e^x, \quad v = 2x$, after integration and differentiation, we get

$u = \int u'dx = \int e^x dx = e^x$, $v' = (2x)' = 2$. The integration by parts formula gives

$\int x^2.e^x dx = e^x.x^2 - \int 2x.e^x dx = e^x.x^2 - e^x.2x + \int 2.e^x dx = x^2e^x - 2x.e^x + 2e^x + C$

From this example, try to remember that most of the time the integration by parts will not be enough to give you the answer after one shot. You may need to do some extra work: another integration by parts or use other techniques.

Example: Evaluate $\int \ln x dx$.

Solution: This is an indefinite integral involving one function. The second needed function is $g(x) = 1$. Since the derivative of this function is 0, the only choice left is to differentiate the other function $f(x) = \ln x$:

$u' = 1,\ v = \ln x,$ after integration and differentiation, we get

$u = \int 1dx = x,\ v' = (\ln x)' = \frac{1}{x}$. The integration by parts formula gives

$\int \ln x dx = x.\ln x - \int x.\frac{1}{x}dx = x.\ln x - \int 1dx = x\ln x - x + C = x(\ln x - 1) + C$.

Remark: Since the derivative of $\ln x$ *is 1/x, it is very common that whenever an integral involves a function which is a product of* $\ln x$ *with another function, to differentiate* $\ln x$ *and integrate the other function.*

Example: Evaluate $\mathrm{I} = \int e^{3x}\cos 2x\, dx$.

Solution: The two functions involved in this example do not exhibit any special behavior when it comes to differentiating or integrating. Therefore, we choose one function to be differentiated and the other one to be integrated. We have

$u' = e^{3x},\ v = \cos 2x$, which implies

$u = \int u'dx = \int e^{3x}dx = \frac{1}{3}e^x,\ v' = (\cos 2x)' = -2\sin 2x$. The integration by parts formula gives

$\mathrm{I} = \int e^{3x}.\cos 2x\, dx = \frac{1}{3}e^{3x}.\cos 2x - \int -\frac{1}{3}e^{3x}.2\sin 2x\, dx = \frac{1}{3}e^{3x}.\cos 2x + \frac{2}{3}\int e^{3x}.\sin 2x\, dx.$

The new integral $\int e^{3x}.\sin 2x\, dx$ is similar in nature to the initial one. One of the common mistake is to do another integration by parts in which we integrate $\sin 2x$ and differentiate e^{3x}. This will simply take you back to your original integral with nothing done. In fact, what you would have done is simply the reverse path of the integration by parts (Do the calculations to convince yourself). Therefore we continue doing another integration by parts as

$u' = e^{3x},\ v = \sin 2x$, which implies

$u = \int u'dx = \int e^{3x}dx = \frac{1}{3}e^x,\ v' = (\sin 2x)' = 2\cos 2x$. The integration by parts formula

$$\int e^{3x}.\sin 2x\, dx = \frac{1}{3}e^{3x}.\sin 2x - \int \frac{1}{3}e^{3x}.2\cos 2x\, dx = \frac{1}{3}e^{3x}.\sin 2x - \frac{2}{3}\int e^{3x}.\cos 2x\, dx.$$

Combining both formulas we get

$$\mathrm{I} = \int e^{3x}.\cos 2x\, dx = \frac{1}{3}e^{3x}.\cos 2x + \frac{2}{3}\left(\frac{1}{3}e^{3x}.\sin 2x - \frac{2}{3}\int e^{3x}.\cos 2x\, dx\right).$$

Easy calculations give

$$\mathrm{I} = \int e^{3x}.\cos 2x\, dx = \frac{1}{3}e^{3x}.\cos 2x + \frac{2}{9}e^{3x}.\sin 2x - \frac{4}{9}\mathrm{I}.$$

After two integration by parts, we get an integral identical to the initial one. You may wonder why and simply because the derivative and integration of e^x are the same while you need two derivatives of the cosine function to generate the same function. Finally easy algebraic manipulation gives

$$\mathrm{I} = \int e^{3x}.\cos 2x\, dx = \frac{3}{13}e^{3x}.\cos 2x + \frac{2}{13}e^{3x}.\sin 2x + C$$

2. DEFINITE INTEGRALS

Integration is vital to many scientific areas. Many powerful mathematical tools are based on integration. Differential equations for instance are the direct consequence of the development of integration.

So what is integration? Integration stems from two different problems. The more immediate problem is to find the inverse transform of the derivative. This concept is known as finding the **antiderivative**. The other problem deals with areas and how to find them. The bridge between these two different problems is the Fundamental Theorem of Calculus.

2.1. The Definite Integral

Definition: Suppose that $F(x)$ is an indefinite integral of $f(x)$, i.e. $F'(x) = f(x)$.

The **Definite Integral** $\int_a^b f(x)\, dx$ where a and b are numbers, is defined to be the number

$$\int_a^b f(x)dx = \left[F(x)\right]_a^b = F(b) - F(a).$$

a and b are called the **Limits of Integration**, a is called the **Lower Limit** and b is called the **Upper Limit**.

Remark: The choice of indefinite integral (choice of constant of integration) does not matter- -the constant of integration cancels out.

$$\int_a^b f(x)dx = \left[F(x)+C\right]_a^b = \left[F(b)+C\right] - \left[F(a)+C\right] = F(b) + C - F(a) - C = F(b) - F(a)$$

Example: Evaluate $\int_2^4 (2x+1)^2\,dx$.

Solution: $\int_2^4 (4x^2+4x+1)dx = \left[4\frac{x^3}{3}+2x^2+x\right]_2^4 = (4\frac{4^3}{3}+2.4^2+4)-(4\frac{2^3}{3}+2.2^2+2) = \frac{302}{3}$.

2.2. The Area Problem and the Definite Integral

Consider the interval <*a*, *b*>. Let $f(x)$ be a continuous function defined on this interval. Let us think once more of the problem of finding the area under the graph of $f(x)$ between $x = a$ and $x = b$.

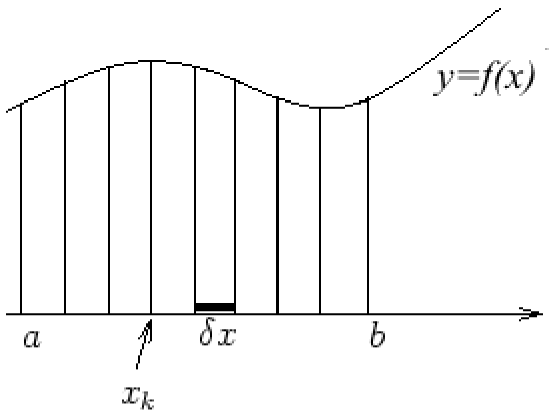
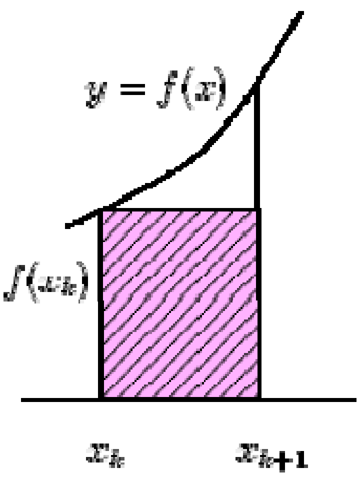


Figure 2.1: Dividing up the area under a curve

We will adopt an approach that is much more elementary (and much older) than our previous method. Divide the interval <*a*, *b*> up into a large number of small parts. For convenience we will take them all to be of the same width, but that is not very important. Now use this subdivision to break up the area into *thin strips* as shown.



Denote the subdivision points by $x_0, x_1, ..., x_n$ where $x_k = a + k\delta x$ and δx is the width of each strip $\delta x = (b - a)/n$.

We can get an approximation to the area under the graph by adding up the areas of the *n* rectangles shown in the picture. The rectangle on the base $\langle x_k, x_{k+1}\rangle$ has area $f(x_k)\delta x$. So

approximate area = $\sum_{k=0}^{n-1} f(x_k)\delta x$

Figure 2.2: One of the rectangles

Now we use the same kind of argument that we used when inventing the derivative. As n gets bigger and bigger we expect the sum of the areas of the rectangles to get closer and closer to the true area under the graph. We would hope that if we took the limit as $n \to \infty$ the sum would tend to the true area as its limiting value. We will assume that this is true. So

$$A = \textit{area under graph} = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\delta x$$

If there exists a number A such that $A = \lim_{n\to\infty} \sum_{k=0}^{n-1} f(x_k)\delta x$, then f is **integrable** on <a, b> and A is the **definite integral** of *f* over <a, b>. This is denoted $\int_a^b f(x)\,dx$.

This interpretation of the definite integral is the one that is most useful in applications, as we will soon see.

2.3. Rules for Definite Integrals

1. Order of Integration: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ $\quad (F(b) - F(a) = -(F(a) - F(b)))$

2. Zero: $\int_a^a f(x)\,dx = 0$

3. Constant Multiple: $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

4. Sum and Difference: $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. Domination: $f(x) \geq g(x)$ on $<a,b> \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

2.4. Integration by Parts in Definite Integrals

Theorem: Let *u(x)* and *v(x)* have continuous derivatives on <a,b>.

Then $\int_a^b u'(x).v(x)\,dx = [u(x).v(x)]_a^b - \int_a^b u(x).v'(x)\,dx$.

Remark: *We usually use the notation* $\int_a^b u'.vdx = [u.v]_a^b - \int_a^b u.v'dx$ *alternatively*

$\int_a^b u.v'dx = [u.v]_a^b - \int_a^b u'.vdx$.

Example: Evaluate $\int_0^{\pi} (x-2)\sin x dx$.

Solution: We choose

$u' = \sin x, \quad v = x-2$, after integration and differentiation, we get

$u = \int u'dx = \int \sin xdx = -\cos x, \; v' = (x-2)' = 1$. The integration by parts formula gives

$\int_0^{\pi} (x-2)\sin xdx = [(x-2)(-\cos x)]_0^{\pi} - \int_0^{\pi} 1.(-\cos x)\,dx = [(2-x)\cos x + \sin x]_0^{\pi} =$

$= (2-\pi)\cos\pi + \sin\pi - (2-0)\cos 0 - \sin 0 = \pi - 2 - 2 = \pi - 4$.

2.5. Substitution in Definite Integrals

In a definite integral $\int_a^b f\ (x)\ dx$ it is always understood that x is independent variable and we are integrating between the limits $x = a$ and $x = b$. Thus when we change to a new independent variable *t*, we must also change limits of integration.

Theorem: Suppose *f* is continuous function and has antiderivative on a interval $<a,b>$ and let function $x = \varphi(t)$ has a continuous derivative on $<\alpha,\beta>$ and φ maps $<\alpha,\beta>$ into $<a,b>$ ($\varphi(\alpha) = a$ and $\varphi(\beta) = b$). Then $\int_a^b f(x)dx = \int_{\alpha}^{\beta} f(\varphi(t))\dot{\varphi}(t)dt$.

In this case, you will never have to go back to the initial variable *x*.

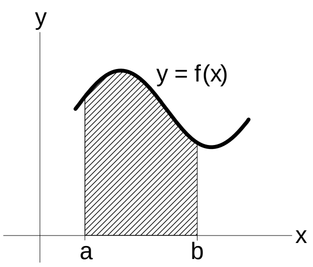
Example: Evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x.\sin xdx$.

Solution: Put $\cos x = t$, so $-\sin\ xdx\ =\ dt$, $\sin\ xdx\ =\ -dt$. As x goes from 0 to $\frac{\pi}{2}$ the value of t goes from α = cos 0 =1 to β = $\cos\frac{\pi}{2}$ = 0. So the integral transforms into :

$$\int_0^{\frac{\pi}{2}} \cos^4 x.\sin xdx\ =\ \int_1^0 t^4(-dt) = -\int_0^1 -t^4 dt = \left[\frac{t^5}{5}\right]_0^1 = \frac{1}{5} - 0 = \frac{1}{5}.$$

2.6. Applications of Integration

Finding Areas under Curves



We want to find the area of the region bounded above by the graph of a positive function $f(x)$, bounded below by the x-axis, bounded to the left by the vertical line $x = a$, and to the right by the vertical line $x = b$ (see Figure 2.3).

Figure 2.3.

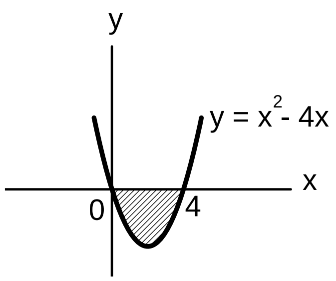
If f is continuous function on <*a, b*> , then the area is given by

$$A = Area\ (shadet) = \int_a^b f\ (x)\ dx$$

Remark: I drew the picture conveniently with the graph above the axis. If f (x) goes negative then the "area" calculated by the integral also goes negative.

Example: Find the area bounded by the curve $y = x^2 - 4x$ and x-axes.

Solution: The area to integrate must be an enclosed area. This time the upper bound is the x-axis, the lower bound is the curve (see Figure 2.4).



We first find a and b by finding the x-coordinates of points of intersection of the function and x-axis. We solve equation $x^2 - 4x = 0$ for x.

$x(x-4) = 0$ so $x = 0,\ x = 4$.

Figure 2.4.

The functional values over the interval <0, 2> are negative. Thus the value of the definite integral over this interval will be negative too. The area is given by

$$A = \int_0^4 \left| x^2 - 4x \right| dx = -\int_0^4 \left(x^2 - 4x \right) dx = -\left[\frac{x^3}{3} - 2x^2 \right]_0^4 = -\left(\frac{64}{3} - 2.16 \right) = \frac{32}{3}.$$

Area Between Two Curves

We have used the definite integral to find the area of the region between a curve and the x-axis over an interval where the curve lies above the x-axis. We can easily extend this technique to finding the area between two curves over an interval where one curve lies above the other (see Figure 2.5).

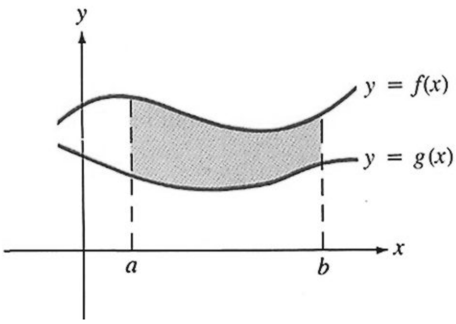


Figure 2.5.

Suppose that the graphs of both $y = f(x)$ and $y = g(x)$ lie above the x-axis, and that the graph of $y = f(x)$ lies above $y = g(x)$ throughout the interval from $x = a$ to $x = b$; that is $f(x) \geq g(x)$ on $<a, b>$.

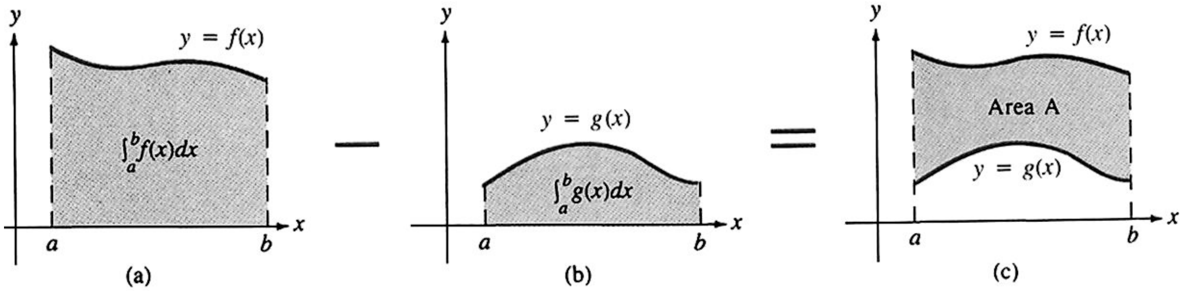


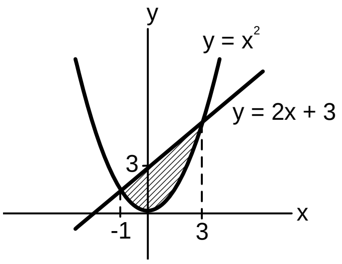
Figure 2.6.

Then $\int_a^b f(x)dx$ gives the area between the graph of $y = f(x)$ and the x-axis (see Figure 2.6 (a)), and $\int_a^b g(x)dx$ gives the area between the graph of $y = g(x)$ and the x-axis (see Figure 2.6 (b)). As Figure 6.9 (c) shows, the area of the region between the graphs of $y = f(x)$ and $y = g(x)$ is the difference of these two areas. That is

$$A = Area\ between\ the\ curves = \int_a^b f\ (x)\ dx - \int_a^b g\ (x)\ dx = \int_a^b \left(f\ (x) - g(x) \right)\ dx\ .$$

Example: Find the area of the region bounded by $y = x^2$ and $y = 2x + 3$.

Solution: The graph of the region is shown in Figure 2.7. We first find *a* and *b* by finding x-coordinates of the points of intersection of the graphs. Setting the y-values equal gives



$x^2 = 2x + 3,$
$x^2 - 2x - 3 = 0,$
$(x - 3)(x + 1) = 0,$
$x = 3, x = -1,$
thus $a = -1$ and $b = 3$.

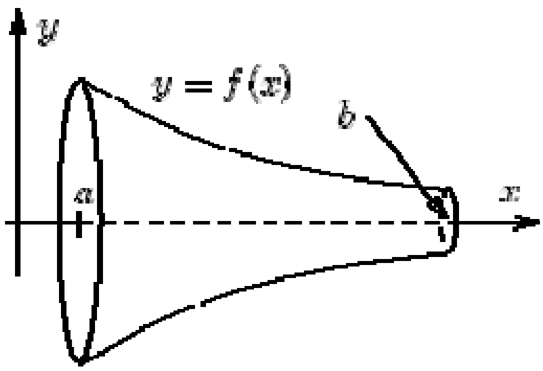
Figure 2.7

The area of enclosed region is:

$$A = \int_{-1}^{3} \left(2x + 3 - x^2\right) dx = \left[x^2 + 3x - \frac{x^3}{3}\right]_{-1}^{3} = (9 + 9 - 9) - \left(1 - 3 + \frac{1}{3}\right) = \frac{32}{3}.$$

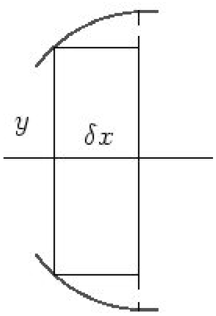
Volumes of Revolution

Take the graph of $y = f(x)$ on the interval $\langle a, b \rangle$ and spin it round the x-axis so as to produce what is known as a **solid of revolution** as shown in Figure 2.8. We want to get a formula for the volume of this solid.



The method is almost exactly the same as in the previous examples. Think of the interval $\langle a, b \rangle$ being subdivided into lots of little bits. Now look at one of the bits and try to get an approximation for the volume of the "thin slice" of the solid obtained by rotating the piece of the graph on this interval.

Figure 2.8.



In the notation of the diagram, the thin slice of the solid is virtually a *cylinder* of radius *y* and thickness δx Figure 2.9. The volume of a cylinder is the product of its height and the area of its base. So we get the approximation $\delta V = \pi y^2 \delta x$ for the volume of the slice.

The approximation to the total volume can then be written as

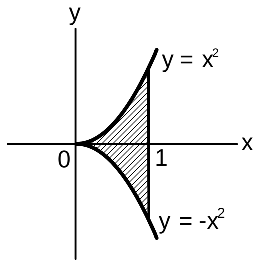
Figure 2.9.

$$V \approx \sum \pi y^2 \delta x \ .$$

Now take the limit as $n \to \infty$ and get

$$V{=}Volume = \pi\int_a^b y^2 dx = \pi\int_a^b \left(f(x)\right)^2 dx \ .$$

Example: Find the volume V of solid generated when the graph of function $y = x^2$ revolves around the x-axis on the interval <0, 1>.



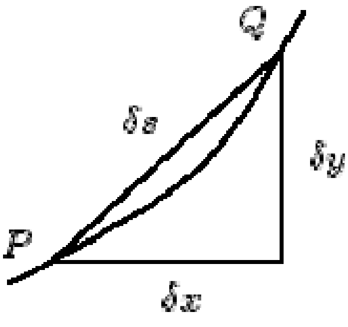
Solution: In Figure 2.10 we see that $a = 0$, $b = 1$. The volume of the solid is

$$V = \pi\int_0^1 \left(x^2\right)^2 dx = \pi\int_0^1 x^4 dx = \pi\left[\frac{x^5}{5}\right]_0^1 = \frac{\pi}{5}.$$

Figure 2.10.

The Length of a Curve

Suppose we want to calculate the length of the graph of $y = f(x)$ between $x = a$ and $x = b$. Subdivide <*a*, *b*> as before into *n* small parts. Look at the graph on one of these parts. The idea is to get an approximation to the length of the graph on this part, in the same way that we used the rectangle approximation when finding the area. Then we add up the approximations and take the limit as $n \to \infty$ so as to produce an integral which gives the true length.



The obvious approach is to use the length of the chord PQ as an approximation to the length of the graph between P and Q. In the notation of Figure 2.11 this length is

$\delta s = \sqrt{\delta x^2 + \delta y^2}$ which we can write more conveniently as

$$\delta s = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2}\,\delta x \ .$$

Figure 2.11.

We now have the approximation to the length which I will write crudely as

$$L \approx \sum \sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\,\delta x \ .$$

Our new interpretation of the definite integral tells us that, as $n \to \infty$ this tends to the value of the definite integral

$$L = Length = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \ .$$

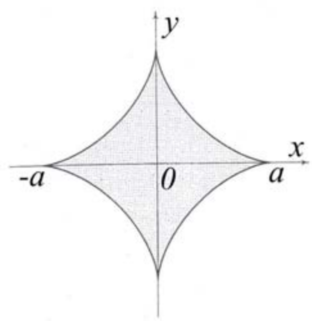
Remark: The $\frac{\delta y}{\delta x}$ *tends to* $\frac{dy}{dx}$ *in the limit.*

Note: If the curve is given parametrically by $x = x(t)$ and $y = y(t)$ then a very similar argument gives us the formula

$$L = Length = \int_\alpha^\beta \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\,dt$$

for the length of the curve between $t = \alpha$ and $t = \beta$.

Example: Find the length L of the asteroid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where $a > 0$.



Solution: Let us employ the parametric expression

$x = a\cos^3 t$, $y = a\sin^3 t$, where $t \in < 0, 2\pi >$.

We calculate $\dot{x} = -3a\cos^2 t \sin t$, $\dot{y} = 3a\sin^2 t \cos t$.

Figure 2.12.

In this case $\sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2} = \sqrt{9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t} = \sqrt{9a^2\cos^2 t\sin^2 t}$

By the symmetry (see Figure 2.11), we have

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{9a^2\cos^2 t\sin^2 t}\,dt = 12a\int_0^{\frac{\pi}{2}} \cos t\sin t\,dt = 6a\int_0^{\frac{\pi}{2}} \sin 2t\,dt = 6a\left[-\frac{\cos 2t}{2}\right]_0^{\frac{\pi}{2}} = 6a \ .$$

Area of Surface of Revolution

I'm going to be brief here and just give you the formula.

Suppose we take the graph $y = f(x)$ on the range <*a*, *b*> and rotate it around the x-axis as before. Then the **Surface Area** of the surface formed by this is given by

$$S = Surface\ Area\ = 2\pi \int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx\ .$$

Example: Find the surface area of the sphere.

Solution: We obtain our sphere by rotating the semicircle. Take the semicircle $y = \sqrt{R^2 - x^2}$ on <- *R, R*> and spin it round the x-axis. We get a sphere of radius *R*.

For this curve $\left(\frac{dy}{dx}\right) = \frac{-x}{\sqrt{R^2 - x^2}}$.

So $\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}$.

So the area is given by

$$S{=}Surface\ of\ a\ sphere{=}\ 2\pi\int_{-R}^{R}\sqrt{R^2 - x^2}\,\frac{R}{\sqrt{R^2 - x^2}}dx = 2\pi\int_{-R}^{R} R\,dx = 2\pi R\left[x\right]_{-R}^{R} = 4\pi R^2 .$$

So a sphere of radius *R* has area $4\pi R^2$.

3. DIFFERENTIAL EQUATIONS

3.1. Introduction

A **differential equation** is an equation for an unknown function, say $y(x)$, which involves derivatives of the function.

For example: $\frac{dy}{dx} = 2x$, $y'' - 2y' + y = \cos x$, $\frac{y''' - 3y''}{2y^4} = x + y'$.

The **order** of a differential equation is the order of the highest derivative occurring in it. In the above examples the orders are 1,2 and 3.

Remark: Technically, these are known as Ordinary Differential Equations (ODE) because the unknown function is a function of one variable. Differential equations involving functions of several variables and their partial derivatives are called Partial Differential Equations (PDE).

Many laws in science and engineering are statements about the relationship between a quantity and the way in which it changes. The change is often measured by a derivative and therefore the mathematical expression of these laws tends to be in terms of differential equations.

Given a differential equation the obvious reaction is to try to solve it for the unknown function. As with integrals, and for much the same reason, this is easier said than done.

Consider the differential equation $y''(x) = 2x$, where the dash denotes differentiation with respect to x. Integrating both sides of this equation with respect to x we get

$$y' = \int 2x dx = x^2 + C \ ,$$

where C is an arbitrary constant of integration. This is one place where it is crucially important to include the constant of integration! Now integrate once more and get

$$y = \int (x^2 + C) dx = \frac{x^3}{3} + Cx + D \ ,$$

where D is a further constant of integration.

We now have the **General Solution** of the differential equation with arbitrary constants C and D. Note that this is really an infinite class of solutions. If we give C and D particular values then we get a **Particular Solution**. For example, $y = \frac{x^3}{3} + x$ and $y = \frac{x^3}{3} + 2x - 5$ are particular solutions of the equation.

The values of the arbitrary constants that we almost invariably acquire when solving a differential equation are usually determined by giving conditions that the solution is required to satisfy. The most common kind of conditions are **Initial Conditions**, where the values of y and some of its derivatives are given for a specific value of x.

Example: Find the solution to $y''(x) = 2x$ then satisfies $y(0) = 1$ and $y'(0) = 0$.

Solution: We know that the general solution is $y = \frac{x^3}{3} + Cx + D$.

The condition $y(0) = 1$ says that $1 = 0 + 0 + D$, so $D = 1$. The condition $y'(0) = 0$ says that $0 = 0 + C$, so $C = 0$. So the required solution is $y = \frac{x^3}{3} + 1$.

Note: An equation of order n generally requires n integrations to get the general solution, so the general solution can be expected to contain n unknown constants and you would expect to have to give n conditions to fix these constants.

3.2. Separable Equations

It is frequently necessary to change the form of a differential equation before it can be solved with the techniques of the previous section. For example, the equation

$y' = y^2$ cannot be solved by simply integrating both sides of the equation with respect to x. We cannot evaluate $\int y^2 dx$ unless we can write y as a function of x, but $y = f(x)$ is the solution we seek.

Because $y' = \dfrac{dy}{dx}$, we can multiply both sides of $\dfrac{dy}{dx} = y^2$ by $\dfrac{dx}{y^2}$ to obtain an equation that has all terms containing y on one side of the equation and all terms containing x on the other side. That is, we obtain

$$\frac{dy}{y^2} = dx .$$

A differential equation is said to be **separable** if it can be manipulated into the form

$$f(y)dy = g(x)dx .$$

The solution of a separable differential equation is obtained by integrating both sides of the equation after the variable have been separated

$$\int f(y)dy = \int g(x)dx + C$$

It may not be possible to express y simply in terms of x.

Example: Solve the differential equation $(x^2 y + x^2)y' = x^3$.

Solution: To write the equation in separable form, we first factor x^2 from the left side and express $y' = \dfrac{dy}{dx}$.

$$x^2(y+1)dy = x^3 dx$$

$$(y+1)dy = \frac{x^3}{x^2}dx$$

The equation is now separated, so we integrate both sides.

$$\int (y+1)dy = \int xdx$$

$$\frac{y^2}{2} + y = \frac{x^2}{2} + C$$

This equation, as well as the equation

$$y^2 + 2y - x^2 = C_1 \quad \text{gives the solution implicitly.}$$

3.3. Linear Differential Equations

Some very important applications are modeled by a special class of differential equations, called **Linear Differential Equations**.

We will solve linear differential equations in which the highest derivative is the first derivative; these are called **First-Order Linear Differential Equations**.

> A first-order **linear differential equation** is an equation of the form
>
> $$y' + p(x)y = q(x)\,,$$
>
> where p and q are functions of x.

Example: Differential equation $y' - \dfrac{y}{x} = e^{\sqrt{x}}$ is called a first-order, linear, nonhomogeneous differential equation. **First-order** – no derivative higher than the first derivative y', **linear** – no powers in y higher than 1, **nonhomogeneous** – $q(x)$ is not zero.

To see how to solve nonhomogeneous differential equation for *y(x)*, let's first consider the simpler equation when $q(x) = 0$, which is called a **Homogeneous Equation**:

Homogeneous Linear Differential Equation

> A differential equation that can be written in the form
>
> $$y' + p(x)y = 0$$
>
> is called a first-order **homogeneous linear differential equation**.

It is understood that x varies over some interval in the real line, and *p(x)* is a continuous function of x in the interval. The equation is called **linear** because y and y' occur only linearly and **homogeneous** because the right side of the equation is zero.

The first order homogeneous linear differential equation has separable variables, because it can be written as

$$\frac{dy}{dx} = -p(x)y \,.$$

This forms allows us to separate all the y-terms on the left and the x-terms on the right:

$$\frac{dy}{y} = -p(x)dx \,,$$

which gives: $$\ln|y| = -\int p(x)dx + c \,,$$

where c is the constant of integration.

Now solve for y $$|y| = e^{-\int p(x)dx + c}\,,$$

$$y(x) = Ce^{-\int p(x)dx}\,,$$

where $C = e^c$ if $y > 0$, and $C = -e^c$ if $y < 0$.

Example: a) Find the general solution of the equation $xy' + 3y = 0$ for $x > 0$.

b) Find the particular solution with the initial value $y(1) = 2$.

Solution: a) We first put the equation into the homogeneous linear form by dividing by x:

$$y' + \frac{3y}{x} = 0 \,,$$

$$\frac{dy}{y} = -3\frac{dx}{x} \,,$$

$$\ln|y| = -3\ln|x| + c = \ln\left|\frac{1}{x^3}\right| + c \,.$$

The constant of integration c is absorbed into the constant C, and the general solution is

$$y(x) = \frac{C}{x^3} \,.$$

b) The particular solution with initial value $y(1) = 2$ is

$$2 = \frac{C}{1}, \quad C = 2\,,$$

$$y(x) = \frac{2}{x^3}\,.$$

Nonhomogeneous Linear Differential Equation

First-order nonhomogeneous linear differential equations are those in which, after isolating the linear terms containing $y(x)$ and $y'(x)$ on the left side of the equation, the right side is ***not*** identically zero. In these cases the right hand side of the equation is usually represented as one function $q(x)$, and the standard form looks like

$$y' + p(x)y = q(x) \ .$$

Remark: When the right hand side is actually a constant k, it is still valid to think of it as a function; it's merely the constant function $q(x) = k$ for all x.

Method for Solving Linear Differential Equation

A general solution of a linear equation can be found by the method called **Variation of Constants**:

1. Start with a solution $\tilde{y}(x)$ of the corresponding homogeneous equation

$$y' + p(x)y = 0 \ ,$$

$$\tilde{y}(x) = Ce^{-\int p(x)dx} = Cu(x) \ ,$$

where $$u(x) = e^{-\int p(x)dx} \ .$$

2. We begin by assuming that the particular solution has the form $y(x) = C(x)u(x)$, where $C(x)$ is an unknown function (we replaced constant C by a function $C(x)$). We substitute this into the differential equation

$$\frac{d}{dx}y + p(x)y = q(x),$$

$$\left(C'(x)u(x) + C(x)u'(x)\right) + p(x)C(x)u(x) = q(x),$$

$$C'(x)u(x) + C(x)\left(u'(x) + p(x)u(x)\right) = q(x) \ .$$

Since $u(x)$ is a solution of homogeneous equation,

$$u'(x) + p(x)u(x) = 0 .$$

We obtain $$C'(x) = \frac{q(x)}{u(x)},$$

$$C(x) = \int \frac{q(x)}{u(x)}dx + K \ ,$$

where K is again the constant of integration.

Thus, the final expression for $y(x)$ is:

$$y(x) = C(x)u(x) = Ku(x) + u(x)\int \frac{q(x)}{u(x)}dx \ .$$

Note that ***the general solution*** $y(x) = \tilde{y}(x) + Y(x)$ *is the sum of an arbitrary constant times* ***a homogeneous solution****,* $\tilde{y}(x) = Ku(x)$*, that satisfies* $y' + p(x)y = 0$ *and* ***a particular solution****,* $Y(x) = u(x)\int \frac{q(x)}{u(x)}dx$*, that satisfies* $Y' + p(x)Y = q(x)$*.*

Example: a) Find the general solution of the equation $xy' + 3y = x^3$ for $x > 0$.

b) Find the particular solution with the initial value $y(1) = 2$.

Solution: a) We first put the equation into the homogeneous linear form by dividing by x:

$$y' + \frac{3y}{x} = x^2 \ .$$

The solution $\tilde{y}(x)$ of the corresponding homogeneous equation

$$y' + \frac{3y}{x} = 0$$

is (see page 22)

$$\tilde{y}(x) = \frac{C}{x^3} .$$

We assume that the general solution has the form $y(x) = \frac{C(x)}{x^3}$ where $C(x)$ is an unknown function. We substitute this into the differential equation

$$\frac{C'(x)x^3 - 3x^2C(x)}{x^6} + \frac{3\frac{C(x)}{x^3}}{x} = x^2 \ ,$$

$$\frac{C'(x)}{x^3} - \frac{3C(x)}{x^4} + \frac{3C(x)}{x^4} = x^2 \ ,$$

$$C'(x) = x^5 \ ,$$

$$C(x) = \int x^5 dx = \frac{x^6}{6} + K \ .$$

The general solution is

$$y(x) = \frac{C(x)}{x^3} = \frac{\frac{x^6}{6} + K}{x^3} = \frac{x^3}{6} + \frac{K}{x^3}$$

b) The particular solution with initial value $y(1)=2$ is

$$2=\frac{1^3}{6}+\frac{K}{1^3}, \quad K=\frac{11}{6} \ .$$

Required particular solution is

$$y(x)=\frac{x^3}{6}+\frac{\frac{11}{6}}{x^3}=\frac{1}{6}(x^3+\frac{11}{x^3}) \ .$$

3.4. Linear Differential Equations nth order

Many systems can be represented in mathematical form using **Linear Differential Equations**. This subject is the main focus of most introductory courses in differential equations. In addition, linear differential equations with constant coefficients can be solved using relatively simple analytical approaches, and the characteristic solutions that are obtained are simple elementary functions (sinusoids, exponentials, etc.) and they give significant insight into the physical behaviour of the systems under study. In the previous chapter we looked at first order differential equations. In this chapter we will move on to nth order differential equations.

A linear differential equation is any differential equation that can be wrote in the following form.

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+...+p_1(x)y'+p_0(x)y=f(x) \ ,$$

where the $p_i(x)$ coefficients and right hand side forcing function, $f(x)$, are continuous functions. For example, the standard form for a second order system is

$$y''+p(x)y'+q(x)y=f(x)$$

Remark: The important thing to note about linear differential equations is that there are no products of the function, $y(x)$, and it's derivatives and neither the function or it's derivatives occur to any power other than the first power.

Initial Conditions is set of conditions on the solution of the form

$$y(x_0)=b_0,\ y'(x_0)=b_1,\ \dots\ ,y^{(n-1)}(x_0)=b_{n-1} \ .$$

The number of initial conditions that are required for a given differential equation will depend upon the order of the differential equation as we will see.

When $f(x) = 0$, the equation is called **homogeneous**, otherwise it is called **nonhomogeneous**. To a nonhomogeneous equation we associate the so called **associated homogeneous equation**:

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \ldots + p_1(x)y' + p_0(x)y = 0 \ .$$

The **general solution** to a problem of this type (i.e. linear ODE) can be written as the sum of a **homogeneous solution** and a **particular solution**,

$$y(x) = \tilde{y}(x) + Y(x) \, ,$$

where $y(x)$ is the general solution, $\tilde{y}(x)$ is the homogeneous solution containing n arbitrary constants, and $Y(x)$ is the particular solution containing no arbitrary constants.

3.5. Homogeneous Equations with Constant Coefficients

When $p_i(x) = a_i =$ constant, the equation is **Equations with Constant Coefficients.** A general constant coefficient homogeneous linear ODE can be written as

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0 \ .$$

One usually assumes a solution of the form $y(x) = e^{\lambda x}$ (for constant λ). Substitution of this expression into the original equation gives

$$\lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1}e^{\lambda x} + \ldots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x} = 0 \ .$$

Dividing by $e^{\lambda x}$ gives the **characteristic equation**

$$\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$$

and the roots of the characteristic polynomial represent values of λ that satisfy the assumed form for $y(x)$. For an nth order system, there will be n roots (not necessarily distinct) to the nth order characteristic equation.

The general solution becomes a linear combination of the individual solutions to the homogeneous equation with the restriction that the n solutions must be linearly independent. For n distinct roots, the homogeneous solution can be written as

$$\tilde{y}(x) = \sum_{i=1}^{n} C_i y_i(x) = \sum_{i=1}^{n} C_i e^{\lambda_i x} \ .$$

The n linearly **independent solutions** form the **basis of solutions** on the interval of interest.

3.6. Homogeneous 2nd Order Equations with Constant Coefficients

A 2nd order constant coefficient homogeneous system can be written as

$$y'' + a_1 y' + a_0 y = 0 \ .$$

The corresponding characteristic equation is

$$\lambda^2 + a_1\lambda + a_0 = 0$$

and the roots of this quadratic equation are given by

$$\lambda_{1,2} = \frac{-a_1 \pm \sqrt{a_1^{\,2} - 4a_0}}{2} \ .$$

With known (distinct) roots, the general solution can be written as

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \ .$$

The actual form of the solution is strongly dependent on whether the roots are real versus complex or distinct versus repeated. In fact three special cases can be identified based on whether the term inside the radical is positive, negative, or zero. These three cases are identified in detail in the remainder of this subsection.

I. Real Distinct Roots:

If $D = a_1^{\,2} - 4a_0 > 0$, the roots λ_1, λ_2 are real and distinct. Therefore the general solution is usually written as above

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \ .$$

II. Repeated Roots:

If $D = a_1^{\,2} - 4a_0 = 0$, one obtains repeated roots. For this situation, the double root is given by $\lambda_{1,2} = \dfrac{-a_1}{2}$.

Therefore, there is only one independent solution, $y_1(x) = e^{-\frac{a_1}{2}x}$. The second independent solution is $y_2(x) = xe^{-\frac{a_1}{2}x}$ and the general solution is

$$y(x) = (C_1 + C_2 x)e^{-\frac{a_1}{2}x} \; .$$

III. Complex Conjugate Roots:

If $D = a_1^{\,2} - 4a_0 < 0$, the roots are complex conjugates. In this case the roots can be written as $\lambda_{1,2} = \alpha \pm \beta i$ with $\alpha = \frac{-a_1}{2}$ and $\beta = \frac{1}{2}\sqrt{|D|}$. The independent solutions are

$y_1(x) = e^{\alpha x} \cos \beta x$ and $y_2(x) = e^{\alpha x} \sin \beta x$. Therefore the general solution is

$$y(x) = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x) \; .$$

Example: Solve the differential equation $y'' + 11y' + 24y = 0$, $y(0) = 0$, $y'(0) = -7$.

Solution: The characteristic equation is $\lambda^2 + 11\lambda + 24 = 0$,

$$(\lambda + 8)(\lambda + 3) = 0 \; .$$

Its roots are $\lambda_1 = -8$ and $\lambda_2 = -3$ and so the general solution is

$$y(x) = C_1 e^{-8x} + C_2 e^{-3x}$$

and its derivative is $y'(x) = -8C_1 e^{-8x} - 3C_2 e^{-3x}$.

Now, plug in the initial conditions to get the following system of equations:

$$y(0) = 0 = C_1 + C_2$$

$$y'(0) = -7 = -8C_1 - 3C_2 \; .$$

Solving this system gives $C_1 = \frac{7}{5}$ and $C_2 = -\frac{7}{5}$. The actual solution to the differential equation is then $y(x) = \frac{7}{5}e^{-8x} - \frac{7}{5}e^{-3x}$.

Example: Solve the differential equation $y''' + 2y'' + y' = 0$.

Solution: The characteristic equation is $\lambda^3 + 2\lambda^2 + \lambda = 0$,

$$\lambda(\lambda^2 + 2\lambda + 1) = \lambda(\lambda + 1)^2 = 0 \; .$$

Its roots are $\lambda_1 = 0$ and the double root $\lambda_{2,3} = -1$ and so the general solution is

$$y(x) = C_1 + C_2 e^{-x} + C_2 x e^{-x} \; .$$

Example: Solve the differential equation $4y''-8y'+5y=0$, $y(0)=0$, $y'(0)=2$.

Solution: The characteristic equation is $4\lambda^2-8\lambda+5=0$

$$\lambda_{1,2}=\frac{-8\pm i\sqrt{|64-80|}}{8}$$

Its roots are $\lambda_{1,2}=1\pm\frac{i}{2}$ and so the general solution is

$y(x)=C_1e^x\cos\frac{x}{2}+C_2e^x\sin\frac{x}{2}=e^x\left(C_1\cos\frac{x}{2}+C_2\sin\frac{x}{2}\right)$ and its derivative is

$$y'(x)=e^x\left(C_1\cos\frac{x}{2}+C_2\sin\frac{x}{2}\right)+e^x\frac{1}{2}\left(-C_1\sin\frac{x}{2}+C_2\cos\frac{x}{2}\right) .$$

Now, plug in the initial conditions to get the following system of equations:

$$y(0)=0=C_1+0C_2 ,$$

$$y'(0)=2=C_1+\frac{1}{2}C_2 .$$

Solving this system gives $C_1=0$ and $C_2=4$. The actual solution to the differential equation is then $y(x)=4e^x\sin\frac{x}{2}$.

3.6. Nonhomogeneous Equations with Constant Coefficients

The n[th] order, linear nonhomogeneous differential equation is

$$y^{(n)}+a_{n-1}y^{(n-1)}+...+a_1y'+a_0y=f(x),$$

where *f(x)* is a non-zero function.

We need to know a set of fundamental solutions $\{y_1,y_2,...,y_n\}$ of the associated homogeneous equation $y^{(n)}+a_{n-1}y^{(n-1)}+...+a_1y'+a_0y=0$.

We know that, in this case, the general solution of the associated homogeneous equation is $\tilde{y}(x)=C_1y_1(x)+C_2y_2(x)+...+C_ny_n(x)$.

The general solution to a linear nonhomogeneous differential equation can be written as the sum of a homogeneous solution and a particular solution,

$$y(x)=\tilde{y}(x)+Y(x) .$$

Method of Variation of Parameters

A more general method for finding particular solutions is the variation of parameter technique. The method can be summarized as follows:

Given the n[th] order, linear nonhomogeneous differential equation with homogeneous solution

$\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x) + \ldots + C_n y_n(x)$.

One can write the particular solution as

$Y(x) = C_1(x) y_1(x) + C_2(x) y_2(x) + \ldots + C_n(x) y_n(x)$,

where $C_1(x),\ C_2(x)\ \ldots\ C_n(x)$ are unknown functions.

Second Order, Linear Nonhomogeneous Differential Equation

We will take a look at the method that can be used to find a particular solution to an equation

$$y'' + a_1 y' + a_0 y = f(x).$$

The general solution to the associated homogeneous differential equation is

$$\tilde{y}(x) = C_1 y_1(x) + C_2 y_2(x), \text{ where}$$

$y_1(x)$ and $y_2(x)$ are a fundamental set of solutions.

What we're going to do is see if we can find a pair of functions, $C_1(x)$ and $C_2(x)$ so that

$$Y(x) = C_1(x) y_1(x) + C_2(x) y_2(x)$$

will be a particular solution to nonhomogeneous differential equation. We have two unknowns here and so we'll need two equations eventually. One equation is easy. Our proposed solution must satisfy the differential equation. The second equation can come from a variety of places. We are going to get our second equation simply by making an assumption that will make our work easier.

So, let's start. If we're going to plug our proposed solution into the differential equation. We're going to need some derivatives so let's get those. The first derivative is

$$Y'(x) = C_1' y_1 + C_1 y_1' + C_2' y_2 + C_2 y_2' \ .$$

Here's the assumption. Simply to make the first derivative easier to deal with we are going to assume that

$$C_1' y_1 + C_2' y_2 = 0 \ .$$

Now, there is no reason ahead of time to believe that this can be done. However, we will see that this will work out. We simply make this assumption on the hope that it won't cause problems down the road and to make the first derivative easier so don't get excited about it. With this assumption the first derivative becomes

$$Y'(x) = C_1 y_1' + C_2 y_2' \ .$$

The second derivative is then

$$Y''(x) = C_1' y_1' + C_1 y_1'' + C_2' y_2' + C_2 y_2'' \ .$$

Plug the solution and it's derivatives into the nonhomogeneous differential equation. Rearranging a little gives

$$C_1' y_1' + C_2' y_2' = f(x) \, .$$

The two equations that we want so solve for the unknown functions are

$$\begin{aligned} C_1' y_1 + C_2' y_2 &= 0 \\ C_1' y_1' + C_2' y_2' &= f(x) \end{aligned}$$

Solving this system is actually quite simple

$$C_1'(x) = \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2} \quad \text{and}$$

$$C_2'(x) = \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} \ .$$

Next, let's notice that $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ is the **Wronskian** of y_1 and y_2. Finally, all that we need to do is integrate $C_1'(x)$ and $C_2'(x)$ in order to determine what $C_1(x)$ and $C_2(x)$ are. Doing this gives

$$C_1(x) = \int \frac{-y_2 f(x)}{y_1 y_2' - y_1' y_2} dx \quad \text{and}$$

$$C_2(x) = \int \frac{y_1 f(x)}{y_1 y_2' - y_1' y_2} dx \ .$$

A particular solution to the differential equation is

$$Y(x)=C_1(x)y_1(x)+C_2(x)y_2(x)=y_1(x)\int\frac{-y_2 f(x)}{y_1y_2'-y_1'y_2}dx+y_2(x)\int\frac{y_1 f(x)}{y_1y_2'-y_1'y_2}dx\ .$$

Example: Solve the differential equation $y''+y=\dfrac{1}{\cos^3 x}$.

Solution: A set of fundamental solutions of the equation $y''+y=0$ is

$\{y_1=\cos x, y_2=\sin x\}$. We seek a particular solution of the form

$Y(x)=C_1(x)\cos x+C_2(x)\sin x$.

We substitute the expression for $Y(x)$ and its derivatives into the inhomogeneous equation. We have a system of linear equations for $C_1'(x)$ and $C_2'(x)$

$$\begin{aligned}&C_1'\cos x+C_2'\sin x=0\\&-C_1'\sin x+C_2'\cos x=\frac{1}{\cos^3 x}\end{aligned}\ .\quad \text{Here is the Wronskian } W(x)=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=1\ .$$

We solve this system using Kramer's rule and get

$C_1'(x)=-\dfrac{\sin x}{\cos^3 x}$ and $C_2'(x)=\dfrac{1}{\cos^2 x}$.

Using techniques of integration, we get

$C_1(x)=-\dfrac{1}{2\cos^2 x}$ and $C_2(x)=\operatorname{tg} x$. The particular solution is

$$Y(x)=-\frac{1}{2\cos^2 x}\cos x+\operatorname{tg} x\sin x=-\frac{1}{2\cos x}+\frac{\sin^2 x}{\cos x}\ .$$

The general solution of the inhomogeneous equation is

$$y(x)=\tilde{y}(x)+Y(x)=C_1\cos x+C_2\sin x-\frac{1}{2\cos x}+\frac{\sin^2 x}{\cos x}\ .$$

Method of Undetermined Coefficients

For some simple differential equations, (primarily constant coefficient equations), and some simple inhomogeneities we are able to guess the form of a particular solution $Y(x)$. This form will contain some unknown parameters. We substitute this form into the differential equation to determine the parameters and thus determine a particular solution.

Consider an n^{th} order inhomogeneous equation with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f(x) \ .$$

We can guess the form of a particular solution, if $f(x)$ is one of a few simple forms

$f(x) = P_n(x)e^{\alpha x}\cos(\beta x)$ or $f(x) = P_n(x)e^{\alpha x}\sin(\beta x)$, where $P_n(x)$ is a polynomial function with degree *n*.

Then a particular solution $Y(x)$ is given by

$Y(x) = x^k \left(Q_n(x)e^{\alpha x}\cos(\beta x) + R_n(x)e^{\alpha x}\sin(\beta x)\right)$ where

$Q_n(x) = A_0 + A_1 x + ... + A_n x^n$, and $R_n(x) = B_0 + B_1 x + ... + B_n x^n$, where the constants A_i and B_i have to be determined. The power k is equal to 0 if $\alpha + i\beta$ is not a root of the characteristic equation. If $\alpha + i\beta$ is a simple root, then $k = 1$ and $k = r$ if $\alpha + i\beta$ is a root of mutiplicity *r*.

Remark: *If the nonhomogeneous term f(x) satisfies the following*

$$f(x) = f_1(x) + f_2(x) + ... + f_N(x) = \sum_{i=1}^{N} f_i(x) \ ,$$

where $f_i(x)$ *are of the forms cited above, then we split the original equation into N equations*

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = f_i(x), \quad i = 1, 2, ..., N \ .$$

Then find a particular solution $Y_i(x)$. *A particular solution to the original equation is given by*

$$Y(x) = Y_1(x) + Y_2(x) + ... + Y_N(x) = \sum_{i=1}^{N} Y_i(x) \ .$$

Example: Find a general solution to the equation $y'' - 3y' - 4y = 3e^{2x} + 2\sin x - 8e^{-x}$

Solution: We split the equation into the following three equations:

(1) $y'' - 3y' - 4y = 3e^{2x}$,

(2) $y'' - 3y' - 4y = 2\sin x$,

(3) $y'' - 3y' - 4y = -8e^{-x}$.

The roots of the characteristic equation $\lambda^2 - 3\lambda - 4 = 0$ are $\lambda = -1$ and $\lambda = 4$.

Particular solution to Equation (1):

Since $\alpha = 2$, and $\beta = 0$, then $\alpha + i\beta = 2$, which is not one of the roots. Then $k = 0$.

The particular solution is given as $Y_1 = Ae^{2x}$

If we plug it into the equation (1), we get

$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x} ,$$

which implies $A = -\frac{1}{2}$, that is, $Y_1 = -\frac{1}{2}e^{2x}$.

Particular solution to Equation (2):

Since $\alpha = 0$, and $\beta = 1$, then $\alpha + i\beta = +i$, which is not one of the roots. Then $k = 0$.

The particular solution is given as $Y_2 = A\cos x + B\sin x$.

If we plug it into the equation (2), we get

$$(-A\cos x - B\sin x) - 3(-A\sin x + B\cos x) - 4(A\cos x + B\sin x) = 2\sin x ,$$

which implies

$\begin{cases} -5A - 3B = 0 \\ 3A - 5B = 2 \end{cases}$, Easy calculations give $A = \frac{3}{17}$ and $B = -\frac{5}{17}$. That is

$$Y_2 = \frac{3}{17}\cos x - \frac{5}{17}\sin x .$$

Particular solution to Equation (3):

Since $\alpha = -1$, and $\beta = 0$, then $\alpha + i\beta = -1$, which is one of the roots. Then $k = 1$.

The particular solution is given as $Y_3 = x^1 Ae^{-x}$.

If we plug it into the equation (3), we get

$$A(x-2)e^{-x} - 3A(-x+1)e^{-x} - 4Axe^{-x} = -8e^{-x} ,$$

which implies $A = \frac{8}{5}$, that is $Y_3 = \frac{8}{5}xe^{-x}$.

A particular solution to the original equation is

$$Y(x) = -\frac{1}{2}e^{2x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x + \frac{8}{5}xe^{-x} .$$

A general solution to the original equation is

$$y(x) = C_1e^{-x} + C_2e^{4x} - \frac{1}{2}e^{2x} + \frac{3}{17}\cos x - \frac{5}{17}\sin x + \frac{8}{5}xe^{-x} .$$

LITERATURE

[1] BUBENÍK, F.: *Mathematics for Engineers.* Vydavatelství ČVUT, Praha1997. ISBN 80-01-01574-2

[2] DEMLOVÁ, M., HAMHALTER, J.: *Calculus* I. Vydavatelství ČVUT, Praha 1998. ISBN 80-01-01110-0

[3] HARSHBARGER, R.J.-REYNOLDS, J.J.: *Calculus with Applications*. D.C.Heath and Company, Lexington1990. ISBN 0-669-21145-1

[4] JAMES, G.: *Modern Engineering Mathematics*. Addison-Wesley, 1992. ISBN 0-201-1805456

[5] JAMES, G.: *Advanced Modern Engineering Mathematics*. Addison-Wesley, 1993. ISBN 0-201-56519-6

[6] PAVELKA, P.-PINKA, P.: *Integrální počet funkcí jedné proměnné*. Matematika IIIa, VŠB-TU Ostrava 1999. ISBN 80-7078-654-X

[7] VLČEK, J.-VRBICKÝ, J.: *Diferenciální rovnice*. Matematika IV, VŠB-TU Ostrava 1997. ISBN 80-7078-438-5