EUROPEAN UNION
European Structural and Investment Funds
Operational Programme Research,
Development and Education

MINISTRY OF EDUCATION,
YOUTH AND SPORTS

Workbook for Mathematics III

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Thanks

The study material was written with the financial support of the project Technology for the Future 2.0, CZ.02.2.69/0.0/0.0/18_058/0010212

ISBN 978-80-248-4730-6 (on-line)
DOI 10.31490/9788024847306

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1.1 Double integral over rectangular domain

As the definite integral of a continuous positive function of one variable represents the area of the region between the graph and the x-axis, **the double integral** of a continuous positive function of two variables represents the volume of the region between the surface defined by the function $z = f(x, y)$ and the xy-plane which contains its domain. We start with rectangular domain

$$D = \left\{ [x, y] \in \mathbb{R}^2 \colon x \in [a, b], \, y \in [c, d] \right\}$$

on the xy-plane according to figure.
We divide interval $[a, b]$, resp. $[c, d]$ by sequences of points

$$a = x_0 < x_1 < x_2 < \ldots < x_m = b,$$

resp.

$$c = y_0 < y_1 < y_2 < \ldots < y_n = d$$

to intervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$, resp. $[y_{j-1}, y_j]$, $j = 1, 2, \ldots, n$. We denote sizes of each component $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.
This way is the whole rectangular domain divided into $m \cdot n$ small rectangles with area $\Delta D_{ij} = \Delta x_i \cdot \Delta y_j$. Now we can choose an arbitrary point $[\xi_i, \eta_j]$ in each rectangle D_{ij} and we can evaluate the volume of a prism with basis D_{ij} and height $z = f(\xi_i, \eta_j)$. The sum of the volumes

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_i, \eta_j) \cdot \Delta x_i \cdot \Delta y_j$$

represents the volume of the body consisted of such prisms over all rectangles D_{ij} if $f(x, y) \geq 0$ on D.

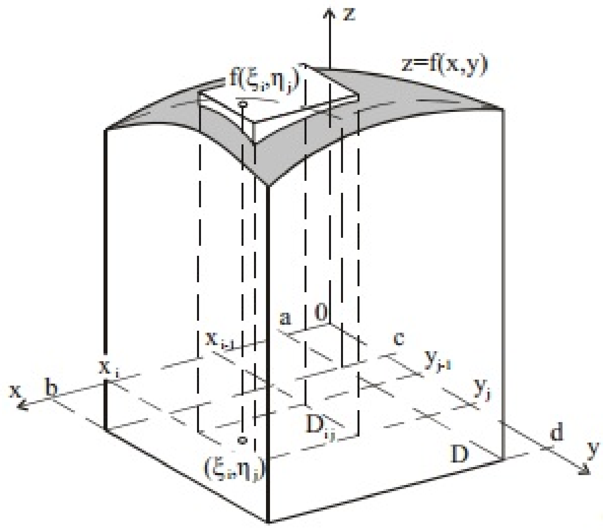
Definition

If there exists

$$\lim \sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_i, \eta_j) \Delta x_i \Delta y_j$$

for $m \to \infty$, $n \to \infty$, $\Delta x_i \to 0$, $\Delta y_j \to 0$ for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we call it **the double integral** of a function $f(x, y)$ over the rectangular domain D and denote it

$$\iint\limits_D f(x, y) \, \mathrm{d}x \, \mathrm{d}y.$$



Theorem (Fubini's theorem)

Let $D = \left\{[x,y] \in \mathbb{R}^2 \colon x \in [a,b]\,,\, y \in [c,d]\right\}$. If function $f(x,y)$ is continuous on rectangle D, then

$$\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_a^b \left(\int\limits_c^d f(x,y)\,\mathrm{d}y \right)\,\mathrm{d}x = \int\limits_c^d \left(\int\limits_a^b f(x,y)\,\mathrm{d}x \right)\,\mathrm{d}y.$$

In fact there are two ways of computing the double integral. If the inner differential is $\mathrm{d}y$ then the limits of the inner integral must have y limits of integration and outer integral must have x limits of integration. We calculate the integral $\int\limits_c^d f(x,y)\,\mathrm{d}y$ by holding x constant and integrating with respect to y as if this were a single integral (similar approach is used for partial derivatives of function of more than one variable). This will result as a function of a single variable x which can be integrated once again. We use similar approach for the second way of computing the double integral. We usually write

$$\int\limits_a^b \left(\int\limits_c^d f(x,y)\,\mathrm{d}y \right)\,\mathrm{d}x = \int\limits_a^b \mathrm{d}x \int\limits_c^d f(x,y)\,\mathrm{d}y$$

and

$$\int\limits_c^d \left(\int\limits_a^b f(x,y)\,\mathrm{d}x \right)\,\mathrm{d}y = \int\limits_c^d \mathrm{d}y \int\limits_a^b f(x,y)\,\mathrm{d}x.$$

Theorem (Properties of the double integral over a rectangular domain)

1. $\iint\limits_D cf(x,y)\,\mathrm{d}x\,\mathrm{d}y = c\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$
2. $\iint\limits_D (f(x,y)+g(x,y))\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_D g(x,y)\,\mathrm{d}x\,\mathrm{d}y,$

3. $\displaystyle\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$

where f, g are continuous functions on D, $c \in \mathbb{R}$ and D_1, D_2 are non-overlapping rectangles that fulfil
$D = D_1 \cup D_2$.

Example 1

Compute $I = \displaystyle\iint\limits_D (2xy + 4x)\,\mathrm{d}x\,\mathrm{d}y$ over the domain D: $0 \le x \le 2$, $-1 \le y \le 3$.

We will show both ways of the computing
a) by integrating the inner integral with respect to variable x

$$I = \iint\limits_D (2xy + 4x)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{-1}^{3} \left(\int\limits_{0}^{2} (2xy + 4x)\,\mathrm{d}x \right) \mathrm{d}y$$

$$= \int\limits_{-1}^{3} \left[x^2 y + 2x^2 \right]_0^2 \mathrm{d}y = \int\limits_{-1}^{3} (4y + 8)\,\mathrm{d}y = \left[2y^2 + 8y \right]_{-1}^{3} = 48$$

b) by integrating the inner integral with respect to variable y

$$I = \int\limits_{0}^{2} \left(\int\limits_{-1}^{3} (2xy + 4x)\,\mathrm{d}y \right) \mathrm{d}x = \int\limits_{0}^{2} \left[xy^2 + 4xy \right]_{-1}^{3} \mathrm{d}x$$

$$= \int\limits_{0}^{2} ((9x + 12x) - (x - 4x))\,\mathrm{d}x = \int\limits_{0}^{2} 24x\,\mathrm{d}x = \left[12x^2 \right]_0^2 = 48$$

If the integrand $f(x,y)$ can be written as a multiplication of two functions of one variable $f(x,y) = f_1(x) \cdot f_2(y)$, then it holds:

$$\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{a}^{b} f_1(x)\,\mathrm{d}x \cdot \int\limits_{c}^{d} f_2(y)\,\mathrm{d}y.$$

Compute the integral by using decomposition on two functions of one variable.

$$I = \iint\limits_D 2x(y + 2)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{2} 2x\,\mathrm{d}x \cdot \int\limits_{-1}^{3} (y + 2)\,\mathrm{d}y$$

$$= \left[x^2 \right]_0^2 \cdot \left[\frac{y^2}{2} + 2y \right]_{-1}^{3} = 4 \cdot \left[\left(\frac{9}{2} + 6 \right) - \left(\frac{1}{2} - 2 \right) \right] = 48$$

Remark

If the decomposition is not possible, we can always use Fubini's theorem.

Example 2

Compute $I = \iint_D x\sqrt{x^2+y}\,\mathrm{d}x\,\mathrm{d}y$ over the domain $D\colon 0 \le x \le 1, 0 \le y \le 3$.

$$I = \int_0^3 \mathrm{d}y \int_0^1 x\sqrt{x^2+y}\,\mathrm{d}x = \left| \begin{array}{ll} t = x^2+y & 0 \to y \\ \mathrm{d}t = 2x\,\mathrm{d}x & 1 \to y+1 \end{array} \right| = \frac{1}{2}\int_0^3 \mathrm{d}y \int_y^{y+1} \sqrt{t}\,\mathrm{d}t$$

$$= \frac{1}{2}\int_0^3 \left[\frac{2}{3}\sqrt{t^3}\right]_y^{y+1} \mathrm{d}y = \frac{1}{3}\int_0^3 \left(\sqrt{(y+1)^3} - \sqrt{y^3}\right) \mathrm{d}y = \frac{1}{3}\left[\frac{2}{5}\sqrt{(y+1)^5} - \frac{2}{5}\sqrt{y^5}\right]_0^3$$

$$= \frac{2}{15}\left(32 - 9\sqrt{3} - 1\right) = \frac{2}{15}\left(31 - 9\sqrt{3}\right).$$

Example 3

Compute $I = \iint_D \left(2x^2y+y^3\right)\cos x\,\mathrm{d}x\,\mathrm{d}y$ over the domain $D\colon 0 \le x \le \frac{\pi}{2}, -1 \le y \le 1$.

Remark

Although generally the order of integration doesn't matter, in some cases the integral can be easily solved by using one way of integration while it can be rather complicated using the other way. Everything depends on the integrand $f(x, y)$ itself and on the limits of integration.

a) First we integrate the inner integral with respect to variable x

$$I = \int_{-1}^{1} \mathrm{d}y \int_0^{\frac{\pi}{2}} \left(2x^2y+y^3\right)\cos x\,\mathrm{d}x = \left| \begin{array}{ll} u = 2x^2y+y^3 & v' = \cos x \\ u' = 4xy & v = \sin x \end{array} \right|$$

$$= \int_{-1}^{1} \left(\left[\left(2x^2y+y^3\right)\sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 4xy\sin x\,\mathrm{d}x \right) \mathrm{d}y = \left| \begin{array}{ll} u = 4xy & v' = \sin x \\ u' = 4y & v = -\cos x \end{array} \right|$$

$$= \int_{-1}^{1} \left(\frac{\pi^2}{2}y + y^3 - \left[-4xy\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} -4y\cos x\,\mathrm{d}x \right) \mathrm{d}y$$

$$= \int_{-1}^{1} \left(\frac{\pi^2}{2}y + y^3 - 4y\left[\sin x\right]_0^{\frac{\pi}{2}} \right) \mathrm{d}y = \int_{-1}^{1} \left(\frac{\pi^2}{2}y + y^3 - 4y \right) \mathrm{d}y$$

$$= \left[\frac{\pi^2}{4}y^2 + \frac{y^4}{4} - 2y^2\right]_{-1}^{1} = \frac{\pi^2}{4} + \frac{1}{4} - 2 - \left(\frac{\pi^2}{4} + \frac{1}{4} - 2\right) = 0.$$

b) Now we integrate the inner integral with respect to variable y

$$I = \int_0^{\frac{\pi}{2}} \mathrm{d}x \int_{-1}^{1} \left(2x^2y+y^3\right)\cos x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}} \left[\left(x^2y^2 + \frac{y^4}{4}\right)\cos x\right]_{-1}^{1} \mathrm{d}x$$

$$= \int_0^{\frac{\pi}{2}} \left(\left(x^2 + \frac{1}{4}\right)\cos x - \left(x^2 + \frac{1}{4}\right)\cos x \right) \mathrm{d}x = \int_0^{\frac{\pi}{2}} 0\,\mathrm{d}x = 0.$$

Exercise 4

Compute following integrals over their domains D.

a) $\displaystyle\iint\limits_D \sqrt{5x+4}\ln y \,\mathrm{d}x\,\mathrm{d}y, \quad D\colon 0 \le x \le 1,\ 1 \le y \le 3$

b) $\displaystyle\iint\limits_D \left(x^2+y^2\right) \mathrm{d}x\,\mathrm{d}y, \quad D\colon -2 \le x \le 0,\ -1 \le y \le 2$

c) $\displaystyle\iint\limits_D \sin(2x+y) \,\mathrm{d}x\,\mathrm{d}y, \quad D\colon 0 \le x \le \pi,\ \frac{\pi}{4} \le y \le \pi$

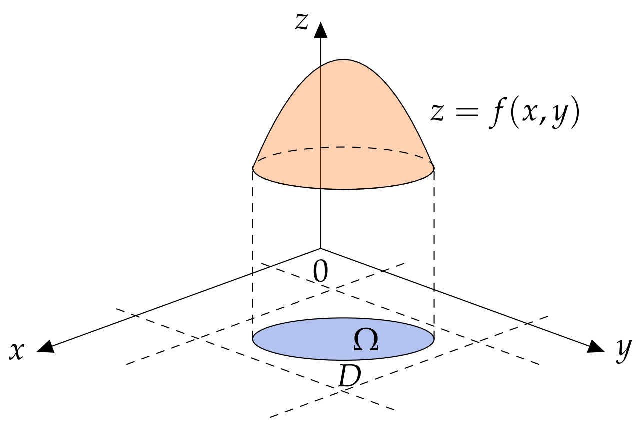
d) $\displaystyle\iint\limits_D \frac{1}{(x+y+1)^2} \,\mathrm{d}x\,\mathrm{d}y, \quad D\colon 0 \le x \le 1, 0 \le y \le 1$

1.2 Double integral over a general domain

There is no reason to limit our problem to rectangular regions. The integral domain can be of a general shape. We extend the Riemann's definition of the double integral over rectangular domain to a closed connected bounded domain Ω without any problem. The domain is connected if we can connect every two points from it by curve that lies within the domain. We can always find a rectangle D that fulfils $\Omega \subseteq D$ and we can define function $f^*(x,y)$ by

$$f^*(x,y) = \begin{cases} f(x,y) & \forall\,[x,y] \in \Omega, \\ 0 & \forall\,[x,y] \in D \setminus \Omega. \end{cases}$$

Then it holds $\displaystyle\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_D f^*(x,y)\,\mathrm{d}x\,\mathrm{d}y.$



The properties of the double integral over a general domain must correspond to next Theorem:

Theorem (Properties of the double integral over a general domain)

1. $\displaystyle\iint\limits_\Omega c f(x,y)\,\mathrm{d}x\,\mathrm{d}y = c\iint\limits_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$

2. $\displaystyle\iint\limits_{\Omega} (f(x,y)+g(x,y))\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\Omega} g(x,y)\,\mathrm{d}x\,\mathrm{d}y,$

3. $\displaystyle\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint\limits_{\Omega_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$

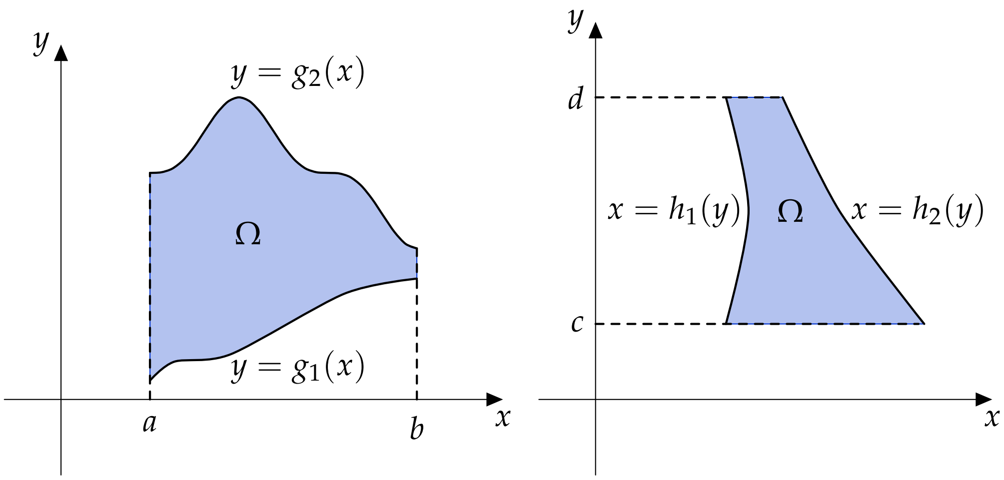
where f, g are continuous functions on Ω, $c \in \mathbb{R}$ and Ω_1, Ω_2 are non-overlapping domains that fulfil $\Omega = \Omega_1 \cup \Omega_2$.

There are two types of domains we need to look at.

Definition

1. **Normal domain with respect to the x-axis** is bounded by lines $x = a$, $x = b$, where $a < b$, and continuous curves $y = g_1(x)$, $y = g_2(x)$, where $g_1(x) < g_2(x)$, for all $x \in [a,b]$.

2. **Normal domain with respect to the y-axis** is bounded by lines $y = c$, $y = d$, where $c < d$, and continuous curves $x = h_1(y)$, $x = h_2(y)$, where $h_1(y) < h_2(y)$, for all $y \in [c,d]$.



Theorem (Fubini's theorem)

1. If the function $f(x,y)$ is continuous on a domain that is normal with respect to the x-axis, then it holds

$$\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_a^b \mathrm{d}x \int\limits_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y.$$

2. If the function $f(x,y)$ is continuous on a domain that is normal with respect to the y-axis, then it holds

$$\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_c^d \mathrm{d}y \int\limits_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x.$$

Example 5

Determine integration limits for $\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ over the domain Ω, which is bounded by curves $y^2 = 2x$ and $x = 2$.

We need to find intersections of curves $y^2 = 2x$ and $x = 2$ by solving the system of these two equations. We can eliminate variable x, receive equation $y^2 = 4$ and solve it. We obtain two solutions $y_1 = 2$, $y_2 = -2$. Given curves intersects each other in points $[2, -2]$ and $[2, 2]$. Treating the domain Ω as a normal with respect to the x-axis, we can see the domain is bounded by $0 \leq x \leq 2$, while limits for variable y must be obtained from the equation $y^2 = 2x$. Therefore $y = \pm\sqrt{2x}$.
We can express inequalities for Ω in the form:

$$\Omega: \quad \begin{aligned} 0 \leq x &\leq 2, \\ -\sqrt{2x} \leq y &\leq \sqrt{2x} \end{aligned}$$

and according to Fubini's theorem

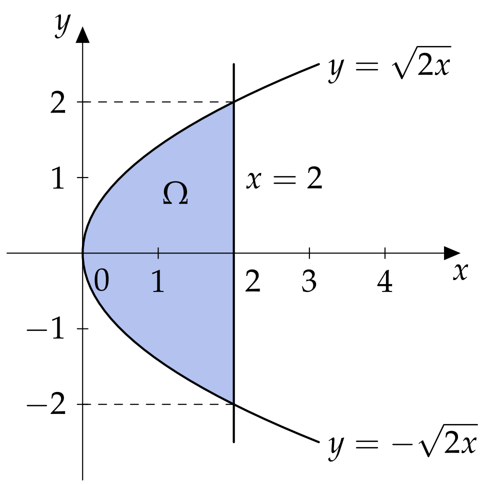
$$\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{2} \mathrm{d}x \int\limits_{-\sqrt{2x}}^{\sqrt{2x}} f(x,y)\,\mathrm{d}y.$$

We can use a similar procedure and express the integral as an integral over normal domain with respect to the y-axis with inequalities

$$\Omega: \quad \begin{aligned} -2 \leq y &\leq 2, \\ \frac{y^2}{2} \leq x &\leq 2. \end{aligned}$$

The double integral then takes form

$$\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{-2}^{2} \mathrm{d}y \int\limits_{\frac{y^2}{2}}^{2} f(x,y)\,\mathrm{d}x.$$



Example 6

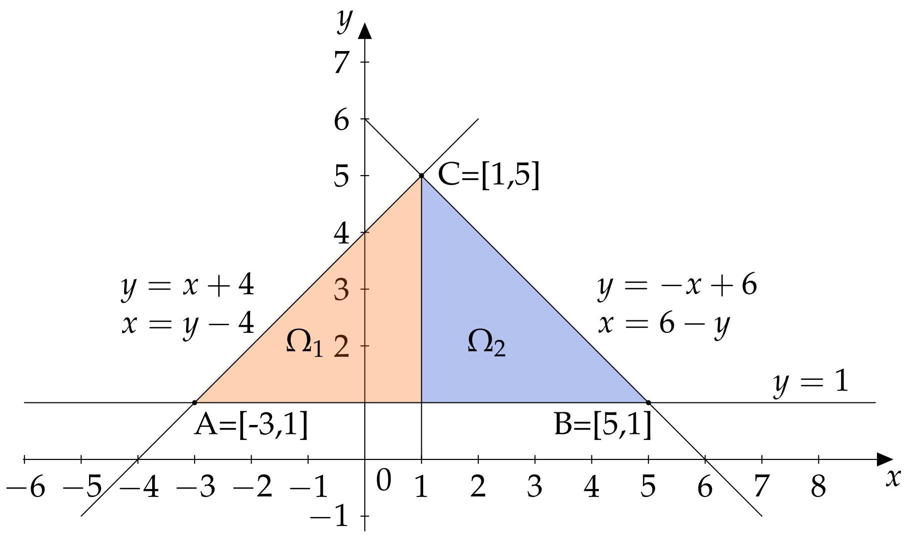
Determine integration limits for $\iint\limits_{\Omega} f(x,y)\ \mathrm{d}x\,\mathrm{d}y$ over the domain Ω, which is a triangle ABC, where $A = [-3,1]$, $B = [5,1]$, $C = [1,5]$.

Remark

Lines can be described algebraically by linear equations $y = ax + b$. We substitute coordinates of points A and C to the equation and we obtain system of two linear equations from which we calculate a and b:

$$\begin{aligned} A:\ 1 &= -3a + b \\ C:\ 5 &= a + b \end{aligned}$$

We get $a = 1$, $b = 4$ and $y = x + 4$.



First, we express the domain as normal with respect to the x-axis. If we bound the domain by $-3 \le x \le 5$, the upper limit of inner integral can't be written as one curve and we need to divide the domain Ω into two subdomains Ω_1, Ω_2 by line $x = 1$:

$$\Omega_1:\ \begin{aligned} -3 \le x &\le 1, \\ 1 \le y &\le x + 4, \end{aligned} \qquad \Omega_2:\ \begin{aligned} 1 \le x &\le 5, \\ 1 \le y &\le 6 - x. \end{aligned}$$

Using Fubini's theorem we can express

$$\iint\limits_{\Omega} f(x,y)\ \mathrm{d}x\,\mathrm{d}y = \int_{-3}^{1} \mathrm{d}x \int_{1}^{x+4} f(x,y)\ \mathrm{d}y + \int_{1}^{5} \mathrm{d}x \int_{1}^{6-x} f(x,y)\ \mathrm{d}y.$$

However, it is much better to express the domain as normal with respect to the y-axis. There is no reason to split the domain which is now bounded by inequalities

$$\Omega:\ \begin{aligned} 1 \le y &\le 5, \\ y - 4 \le x &\le 6 - y, \end{aligned}$$

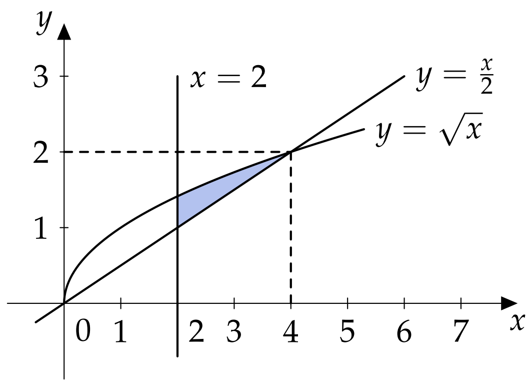
where we have expressed a variable x from boundary equations. The integral is written in the form

$$\iint_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{1}^{5} \mathrm{d}y \int_{y-4}^{6-y} f(x,y)\,\mathrm{d}x.$$

Example 7

Compute $\iint_{\Omega} xy\,\mathrm{d}x\,\mathrm{d}y$, Ω is bounded by $y = \dfrac{x}{2}$, $y = \sqrt{x}$, $x \geq 2$.

Solving the system of equations $y = \dfrac{x}{2}$, $y = \sqrt{x}$ we receive intersections of both curves in $x = 0, x = 4$.



It is better to express the domain as normal with respect to x-axis with boundaries

$$\Omega:\quad \begin{aligned} 2 \leq x &\leq 4, \\ \frac{x}{2} \leq y &\leq \sqrt{x} \end{aligned}$$

and compute the integral

$$\iint_{\Omega} xy\,\mathrm{d}x\,\mathrm{d}y = \int_{2}^{4} \mathrm{d}x \int_{x/2}^{\sqrt{x}} xy\,\mathrm{d}y = \int_{2}^{4} x \left[\frac{y^2}{2}\right]_{x/2}^{\sqrt{x}} \mathrm{d}x = \int_{2}^{4} \left(\frac{x^2}{2} - \frac{x^3}{8}\right) \mathrm{d}x = \left[\frac{x^3}{6} - \frac{x^4}{32}\right]_2^4 = \frac{11}{6}.$$

The second approach requires splitting the domain into two subdomains. It is a good exercise to compute the example this way.

Exercise 8

Compute following integrals over their domains Ω.

a) $\iint_{\Omega} \left(5x^2 - 2xy\right) \mathrm{d}x\,\mathrm{d}y$, $\quad\Omega$ is triangle ABC, where $A = [0,0]$, $B = [2,0]$, $C = [0,1]$

b) $\iint_{\Omega} x^2\,\mathrm{d}x\,\mathrm{d}y$, $\quad\Omega\colon y = \dfrac{16}{x}$, $y = x$, $x = 8$

c) $\iint_{\Omega} 6xy\,\mathrm{d}x\,\mathrm{d}y$, $\quad\Omega\colon y = 0$, $x = 2$, $y = x^2$

d) $\displaystyle\iint_{\Omega} xy\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon\ x^2+4y^2 \le 4,\ x \ge 0,\ y \ge 0$

e) $\displaystyle\iint_{\Omega} (1-2x-3y)\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon\ x^2+y^2 \le 2$

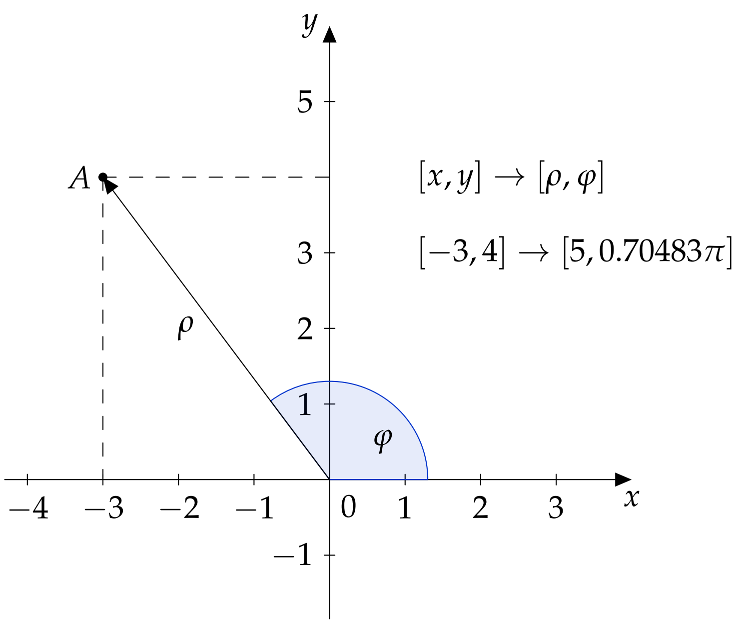
1.3 Double integral in polar coordinates

At this moment we are able to compute the double integral over a general domain. In this section we want to look at some domains that are easier to describe in a terms of polar coordinates. We might have a domain that is a disc, ring or part of a disc or ring. Let us consider a double integral of an arbitrary function over the disc with the center in the origin of coordinates and with the radius $r=2$ (same domain that is used in last Exercise). Using Cartesian coordinates we obtain limits of the integral

$$\Omega\colon \quad \begin{aligned} -2 \le x \le 2&, \\ -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}&. \end{aligned}$$

and by Fubini's theorem the integral can be written in the form

$$\iint_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{-2}^{2} \mathrm{d}x \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,\mathrm{d}y.$$



In such cases using Cartesian coordinates can be tedious. However, we are able to replace Cartesian coordinates x, y by polar coordinates ρ, φ, where ρ denotes a distance between the point $[x,y]$ and the origin of coordinates and is called a **radius**, and, φ denotes the positively oriented angle between positive part of the x-axis and the radius vector and is called **angular coordinate** or **azimuth**.

The transformation to cylindrical coordinates is given by transformation equations

$$\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi. \end{aligned}$$

Transformation to polar coordinates is a special case of mapping region Ω onto Ω^* that is an image of Ω in polar coordinates in our case. For example a disc with the center in the origin of coordinates and with the radius $r = 2$,

$$\Omega = \left\{[x, y] \colon x^2 + y^2 \leq 4\right\},$$

is mapped onto

$$\Omega^* = \{[\rho, \varphi] \colon \rho \in (0, 2], \varphi \in [0, 2\pi)\}.$$

Theorem (Transformation to general coordinates)

- Let equations $x = u(r, s), y = v(r, s)$ map the region Ω bijectively to the region Ω^*.
- Let function $f(x, y)$ be continuous and bounded on Ω and functions $x = u(r, s)$, $y = v(r, s)$ have continuous partial derivatives on $\hat{\Omega}$ that fulfils $\Omega^* \subset \hat{\Omega}$.
- Let $J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix} \neq 0$ in Ω^*.

Then

$$\iint\limits_{\Omega} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega^*} f(u(r, s), v(r, s))\,|J(u, v)|\,\mathrm{d}r\,\mathrm{d}s.$$

Determinant

$$J(u, v) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} \end{vmatrix}$$

is called **Jacobian** or **Jacobi determinant**.
We will use this theorem for transformation of the double integral to polar coordinates as well as the triple integral to cylindrical and spherical coordinates. According to the theorem we replace square element $\mathrm{d}x\,\mathrm{d}y$ by $|J|\ \mathrm{d}\rho\ \mathrm{d}\varphi$, where the Jacobian of the transformation to polar coordinates satisfies

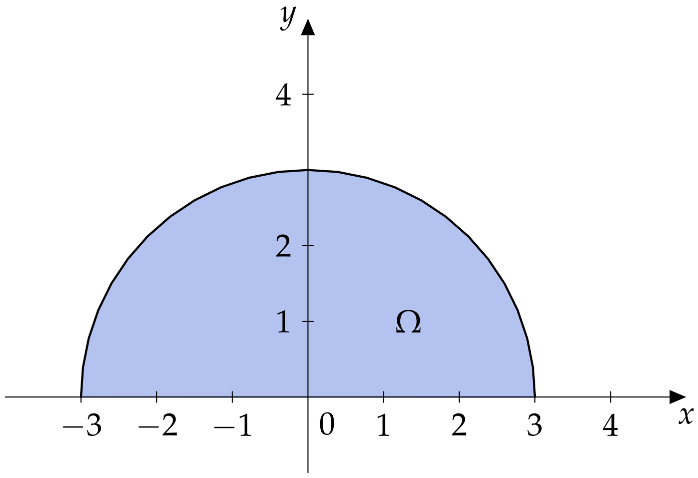
$$J(\rho, \varphi) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho.$$

The transformation of the double integral to polar coordinates can be written in the form

$$\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega^*} f(\rho\cos\varphi, \rho\sin\varphi)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi.$$

Example 9

Compute $\iint\limits_{\Omega} y\,\mathrm{d}x\,\mathrm{d}y$ over the domain $\Omega = \{[x,y]\colon x^2+y^2 \le 9,\ y \ge 0\}$ using transformation to polar coordinates.



The domain Ω is an upper half of the disc with the center in the origin of coordinates and with radius $r = 3$. We use transformation to polar coordinates and obtain the domain

$$\Omega^*\colon \quad \begin{aligned} &0 < \rho \le 3, \\ &0 \le \varphi \le \pi. \end{aligned}$$

We have

$$\iint\limits_{\Omega} y\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega^*} \rho\sin\varphi \cdot \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi = \int\limits_0^3 \rho^2\,\mathrm{d}\rho \cdot \int\limits_0^\pi \sin\varphi\,\mathrm{d}\varphi$$

$$= \left[\frac{\rho^3}{3}\right]_0^3 \cdot [-\cos\varphi]_0^\pi = 18.$$

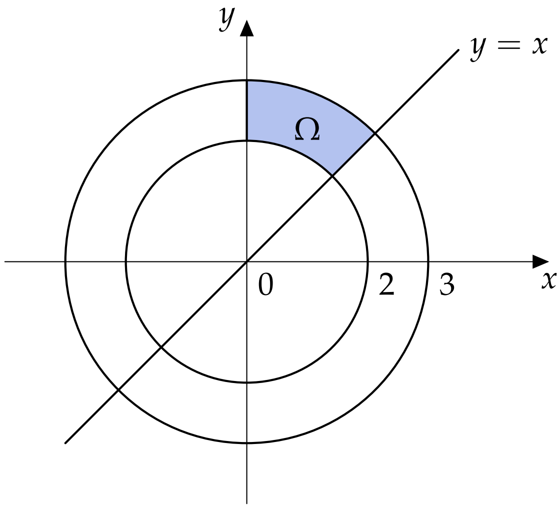
Example 10

Compute $\iint\limits_{\Omega} x\,\mathrm{d}x\,\mathrm{d}y$ over the domain $\Omega = \{[x,y]\colon 4 \le x^2+y^2 \le 9, y \ge x,\ x \ge 0\}$.

We can see real advantage of the transformation on this domain. While using Cartesian coordinates would be complicated, domain

$$\Omega^* = \left\{[\rho,\varphi] : \rho \in [2,3]\,,\ \varphi \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]\right\}$$

for polar coordinates is rectangular.

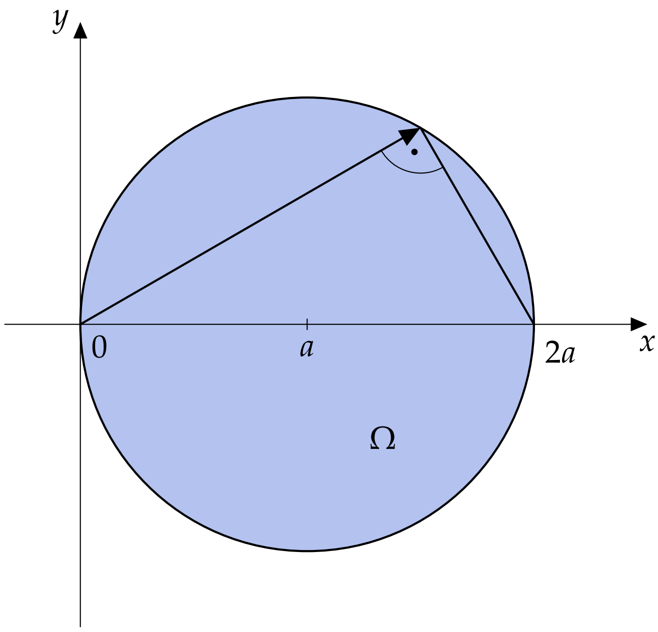


$$\iint\limits_{\Omega} x \,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega^*} \rho\cos\varphi\cdot\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi = \int\limits_{2}^{3}\rho^2\,\mathrm{d}\rho\cdot\int\limits_{\pi/4}^{\pi/2}\cos\varphi\,\mathrm{d}\varphi$$

$$= \left[\frac{\rho^3}{3}\right]_2^3\cdot[\sin\varphi]_{\pi/4}^{\pi/2} = \frac{19}{3}\left(1-\frac{\sqrt{2}}{2}\right)$$

Example 11

Calculate limits of the integral transformed to polar coordinates for the domain $\Omega = \left\{[x,y]\colon x^2+y^2 \le 2ax\right\}$.



First, we find the center and radius of the disc.

$$\begin{aligned} x^2+y^2 &\le 2ax \\ x^2-2ax+a^2+y^2 &\le a^2 \\ (x-a)^2+y^2 &\le a^2 \end{aligned}$$

We have found that center $S=[a,0]$ and radius $r=a$.

Remark

Although, limits of φ are usually between $0 \leq \varphi \leq 2\pi$, in such cases, we use negative limits, to prevent splitting of the domain.

The azimuth must fulfil $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. We can see that the upper limit of coordinate ρ depends on the azimuth φ. We obtain the value of the limit by substituting transformation equations to boundary equations of Ω.

$$\begin{aligned} x^2 + y^2 &= 2ax \\ \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi &= 2a\rho \cos \varphi \\ \rho^2 &= 2a\rho \cos \varphi \\ \rho(\rho - 2a \cos \varphi) &= 0 \end{aligned}$$

Roots $\rho_1 = 0$ and $\rho_2 = 2a \cos \varphi$ are limits of the integral. However, it is necessary to realise the dependency of coordinate ρ on coordinate φ. We can't calculate integrals over such domains as in case of rectangular ones. We need to use Fubini's theorem. The integral of an arbitrary function can be written as

$$\iint\limits_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{-\pi/2}^{\pi/2} \mathrm{d}\varphi \int\limits_{0}^{2a\cos\varphi} f(\rho\cos\varphi, \rho\sin\varphi)\rho\,\mathrm{d}\rho.$$

Exercise 12

Compute following integrals over their domains Ω.

a) $\displaystyle\iint\limits_{\Omega} (1 - 2x - 3y)\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon x^2 + y^2 \leq 2$

b) $\displaystyle\iint\limits_{\Omega} \sqrt{1 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon x^2 + y^2 \leq 1,\ x \geq 0,\ y \geq 0$

c) $\displaystyle\iint\limits_{\Omega} \sin\sqrt{x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon \pi^2 \leq x^2 + y^2 \leq 4\pi^2$

d) $\displaystyle\iint\limits_{\Omega} \frac{\ln\left(x^2 + y^2\right)}{x^2 + y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon 1 \leq x^2 + y^2 \leq \mathrm{e}$

e) $\displaystyle\iint\limits_{\Omega} \sqrt{4 - x^2 - y^2}\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon x^2 + y^2 \leq 2x$

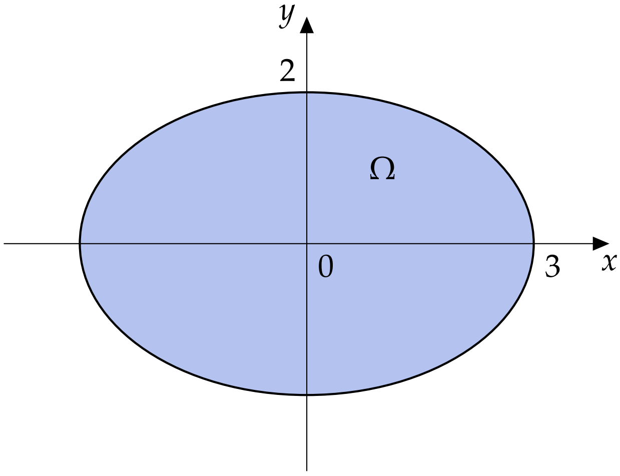
f) $\displaystyle\iint\limits_{\Omega} xy\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon x^2 + y^2 \leq 4y,\ y \geq x \geq 0$

1.4 Double integral in generalized polar coordinates

Example 13

Compute $\displaystyle\iint\limits_{\Omega} \sqrt{4 - \frac{x^2}{9} - \frac{y^2}{4}}\, \mathrm{d}x\, \mathrm{d}y$ over $\Omega = \left\{[x, y] : 4x^2 + 9y^2 \leq 36\right\}$ using transformation to generalized polar coordinates.

The boundary of the domain can be written in the form $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. Therefore, the domain is ellipse with center in the origin of coordinates and semi-axis $a = 3$, $b = 2$.



In such case we use **generalized polar coordinates** in the form

$$\begin{aligned} x &= a\rho\cos\varphi, \\ y &= b\rho\sin\varphi. \end{aligned}$$

For Jacobian of the transformation we obtain

$$J(\rho, \varphi) = \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \varphi} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} a\cos\varphi & -a\rho\sin\varphi \\ b\sin\varphi & b\rho\cos\varphi \end{vmatrix} = ab\rho.$$

Using generalized polar coordinates we obtained transformed domain

$$\Omega^* = \{[\rho, \varphi] : \rho \in (0, 1], \varphi \in [0, 2\pi)\}$$

and we can solve the integral now.

$$\iint\limits_{\Omega} \sqrt{4 - \frac{x^2}{9} - \frac{y^2}{4}}\, \mathrm{d}x\, \mathrm{d}y = \iint\limits_{\Omega^*} \sqrt{4 - \frac{(3\rho\cos\varphi)^2}{9} - \frac{(2\rho\sin\varphi)^2}{4}}\, 6\rho\, \mathrm{d}\rho\, \mathrm{d}\varphi$$

$$= 6\iint\limits_{\Omega^*} \sqrt{4 - \rho^2}\rho\, \mathrm{d}\rho\, \mathrm{d}\varphi = 6\int_0^{2\pi} \mathrm{d}\varphi \cdot \int_0^1 \sqrt{4 - \rho^2}\rho\, \mathrm{d}\rho = 6 \cdot 2\pi \cdot \frac{1}{3}\left(8 - 3\sqrt{3}\right)$$

$$= 4\pi\left(8 - 3\sqrt{3}\right)$$

Integral over coordinate ρ was calculated using substitution

$$\begin{aligned} 4 - \rho^2 &= t, \\ -2\rho\, \mathrm{d}\rho &= \mathrm{d}t. \end{aligned}$$

Exercise 14

Compute following integrals over their domains Ω.

a) $\displaystyle\iint\limits_{\Omega} (2x+y)\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon 4x^2+y^2 \le 16,\ y \le 0,\ x \le 0$

b) $\displaystyle\iint\limits_{\Omega} xy\,\mathrm{d}x\,\mathrm{d}y, \quad \Omega\colon x^2+4y^2 \le 4,\ x \ge 0,\ y \ge 0$

1.5 Practical applications of the double integral

1.5.1 Area of a region

The **area of a region** Ω is given by

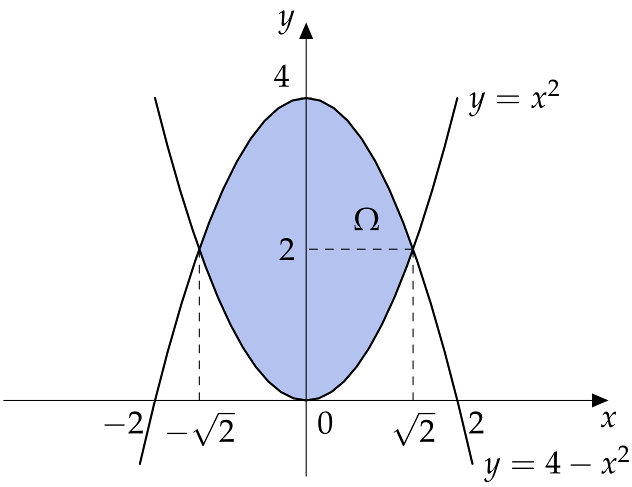
$$A = \iint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y.$$

Example 15

Calculate the area of a region Ω bounded by curves $y = x^2$, $y = 4 - x^2$.

We need to find intersections of both parabolas $y = x^2$, $y = 4 - x^2$ that are $x = \pm\sqrt{2}$. We write the domain as a normal with respect to the x-axis with inequalities in the form:

$$\Omega\colon \quad -\sqrt{2} \le x \le \sqrt{2}, \\ x^2 \le y \le 4 - x^2.$$



We compute the area of our region using symmetry of the domain with respect to the y-axis

$$A = \iint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y = \int\limits_{-\sqrt{2}}^{\sqrt{2}} \mathrm{d}x \int\limits_{x^2}^{4-x^2} \mathrm{d}y$$

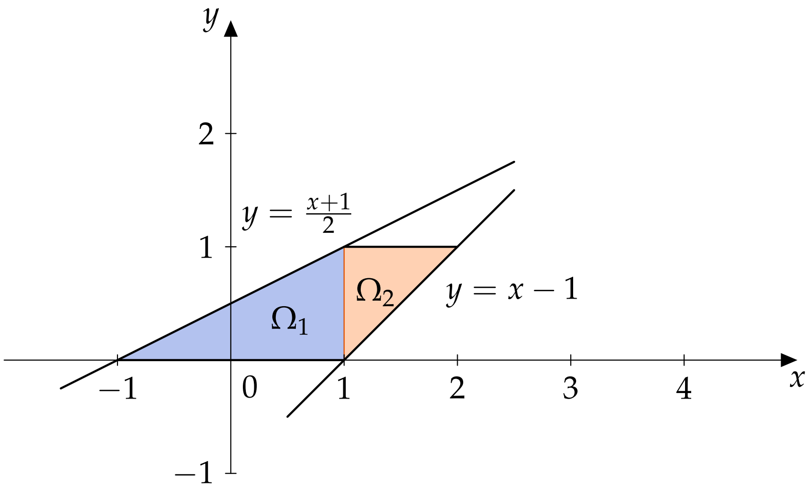
$$= 2\int_0^{\sqrt{2}} \left(4 - 2x^2\right)\, \mathrm{d}x = 2\left[4x - \frac{2}{3}x^3\right]_0^{\sqrt{2}} = \frac{16\sqrt{2}}{3}.$$

Example 16

Compute the area of domain

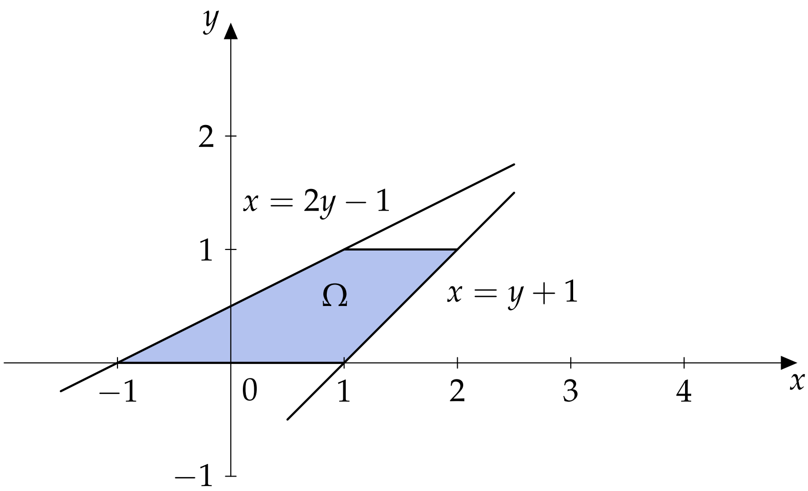
$$\Omega = \{[x, y] : x - y - 1 \le 0,\ x - 2y + 1 \ge 0,\ 0 \le y \le 1\}.$$

Domain is bounded by the lines $y = 0$, $y = 1$, $y = x - 1$ and $y = \dfrac{x+1}{2}$. If we write the domain as a normal with respect to the x-axis, we have to split the domain. It is a good exercise to compute the example in this way.



We write the domain as a normal with respect to the y-axis with inequalities in the form:

$$\Omega: \quad 2y - 1 \le x \le y + 1, \\ 0 \le y \le 1$$



and compute the area of Ω:

$$A = \iint_\Omega \mathrm{d}x\, \mathrm{d}y = \int_0^1 \mathrm{d}x \int_{2y-1}^{y+1} \mathrm{d}y = \int_0^1 (2 - y)\, \mathrm{d}y = \left[2y - \frac{y^2}{2}\right]_0^1 = \frac{3}{2}.$$

Exercise 17

Compute the areas of the regions bounded by curves.

a) $y = x,\ y = 5x,\ x = 1$

b) $y = x^2 - 8x + 12,\ y = -2x + 4$

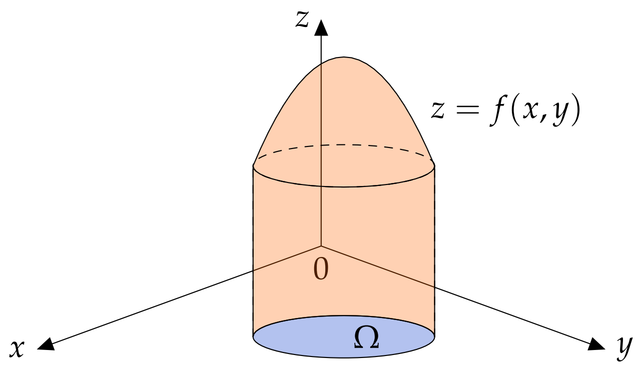
c) $y = 2^x,\ y = 2^{-2x},\ y = 4$

d) $x^2 + y^2 = 4,\ x^2 + y^2 = 4y$

1.5.2 Volume of a body

The **volume of the cylindrical body with basis** Ω bounded by an arbitrary function $f(x,y)$ is given by

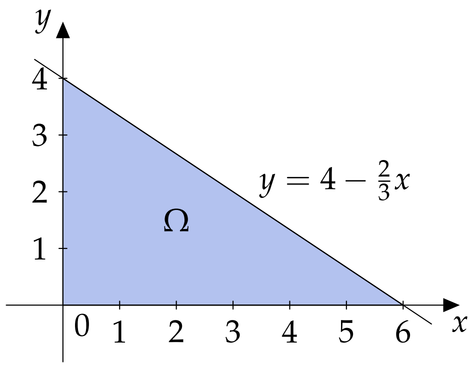
$$V = \iint_{\Omega} |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.$$



Example 18

Calculate the volume of the body bounded by surfaces $2x + 3y = 12$, $2z = y^2$, $x = 0$, $y = 0$, $z = 0$.

The basis of the body lies in the plane $z = 0$. Planes $2x + 3y = 12$, $x = 0$ and $y = 0$ are perpendicular to the basis, thus they define the triangular domain Ω.



Because $z = \dfrac{y^2}{2} \geq 0$ for all $[x,y] \in \Omega$, therefore the surface $z = \dfrac{y^2}{2}$ bounds the body from above. We write the domain as a normal with respect to the x-axis with inequalities for Ω

in the form:

$$\Omega: \quad 0 \le x \le 6,$$
$$0 \le y \le 4 - \frac{2}{3}x.$$

We compute the volume of the body

$$V = \iint_{\Omega} \frac{y^2}{2}\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{2}\int_0^6 \mathrm{d}x \int_0^{4-\frac{2}{3}x} y^2\,\mathrm{d}y = \frac{1}{2}\int_0^6 \left[\frac{y^3}{3}\right]_0^{4-\frac{2}{3}x} \mathrm{d}x$$

$$= \frac{1}{6}\int_0^6 \left(4 - \frac{2}{3}x\right)^3 \mathrm{d}x = \frac{1}{6}\cdot\left(-\frac{3}{2}\right)\left[\frac{\left(4-\frac{2}{3}x\right)^4}{4}\right]_0^6 = 16.$$

Exercise 19

Calculate the volumes of the bodies bounded by given surfaces.

a) $x = 0,\ y = 0,\ z = 0,\ 6x + 3y + z - 12 = 0$

b) $z = 0,\ z = xy,\ y = 0,\ y = \sqrt{x},\ x + y = 2$

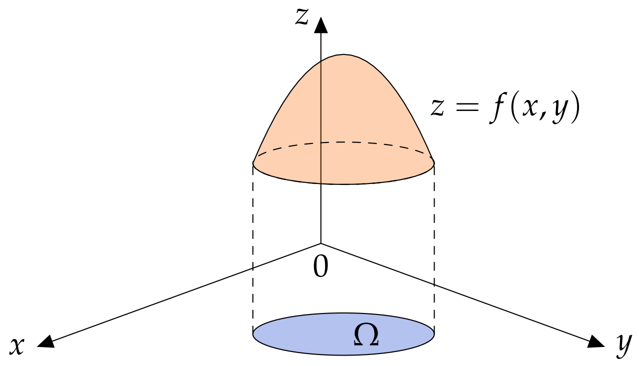
a) $z = 0,\ 2y = x^2,\ z = y^2 - 4$

b) $z = 0,\ z = 1 - x^2 - y^2$

1.5.3 Surface area

We are able to compute the **area of the surface** $z = f(x,y)$ where $[x,y]$ is a point in the region Ω. Function $z = f(x,y)$ must have continuous partial derivatives on Ω. In this case surface area is given by

$$S = \iint_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y.$$



Example 20

Calculate the area of a surface $z = \sqrt{2xy}$ bounded by planes $x = 1$, $x = 2$, $y = 1$ and $y = 4$.

Partial derivatives of z are $\dfrac{\partial z}{\partial x} = \dfrac{y}{\sqrt{2xy}}$ and $\dfrac{\partial z}{\partial y} = \dfrac{x}{\sqrt{2xy}}$. The domain Ω is a rectangle given by inequalities

$$\begin{aligned}\Omega\colon \quad & 1 \le x \le 2,\\ & 1 \le y \le 4\end{aligned}$$

and we calculate the surface area

$$S = \iint\limits_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega} \sqrt{1 + \frac{y^2}{2xy} + \frac{x^2}{2xy}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \iint\limits_{\Omega} \sqrt{\frac{2xy + x^2 + y^2}{2xy}}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega} \sqrt{\frac{(x+y)^2}{2xy}}\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_{\Omega} \frac{x+y}{\sqrt{2xy}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_1^2 \mathrm{d}x \int_1^4 \left(\sqrt{\frac{x}{2}} \cdot y^{-\frac{1}{2}} + \frac{1}{\sqrt{2x}} \cdot y^{\frac{1}{2}}\right) \mathrm{d}y = \int_1^2 \left[\sqrt{\frac{x}{2}} \cdot 2y^{\frac{1}{2}} + \frac{1}{\sqrt{2x}} \cdot \frac{2}{3} y^{\frac{3}{2}}\right]_1^4 \mathrm{d}x$$

$$= \int_1^2 \left(\sqrt{\frac{x}{2}} \cdot 4 + \frac{1}{\sqrt{2x}} \cdot \frac{16}{3} - \sqrt{\frac{x}{2}} \cdot 2 - \frac{1}{\sqrt{2x}} \cdot \frac{2}{3}\right) \mathrm{d}x = \int_1^2 \left(\sqrt{\frac{x}{2}} \cdot 2 + \frac{1}{\sqrt{2x}} \cdot \frac{14}{3}\right) \mathrm{d}x$$

$$= \int_1^2 \left(\frac{2}{\sqrt{2}} \cdot x^{\frac{1}{2}} + \frac{14}{3\sqrt{2}} \cdot x^{-\frac{1}{2}}\right) \mathrm{d}x = \left[\frac{2}{\sqrt{2}} \cdot \frac{2}{3} x^{\frac{3}{2}} + \frac{14}{3\sqrt{2}} \cdot 2x^{\frac{1}{2}}\right]_1^2$$

$$= \frac{8}{3} + \frac{28}{3} - \frac{4}{3\sqrt{2}} - \frac{28}{3\sqrt{2}} = 12 - \frac{32}{3\sqrt{2}} = 12 - \frac{16}{3}\sqrt{2}.$$

Exercise 21

Compute the areas of the surfaces given by:

a) $x + y + z = 4$ bounded by planes $x = 0$, $x = 2$, $y = 0$, $y = 2$,

b) $y^2 + z^2 = 9$ bounded by planes $x = 0$, $x = 2$, $y = -3$, $y = 3$,

c) $z = xy$ in the cylinder $x^2 + y^2 = 4$.

1.5.4 Center of mass

Let $\sigma(x, y) > 0$ be a surface density defined for each $[x, y] \in \Omega$. **The mass** of a domain Ω is defined by

$$m = \iint\limits_{\Omega} \sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

Static moment of the domain Ω with respect to the x-axis resp. y-axis is given by

$$S_x = \iint\limits_{\Omega} y\,\sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y \qquad \text{resp.} \qquad S_y = \iint\limits_{\Omega} x\,\sigma(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

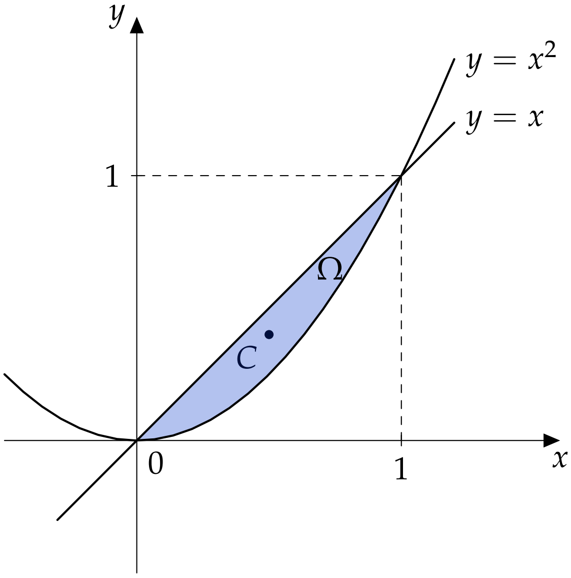
The coordinates of the **center of mass** $C = [\xi, \eta]$ can be expressed as

$$\xi = \frac{S_y}{m}, \qquad \eta = \frac{S_x}{m}.$$

Example 22

Compute the coordinates of the center of mass of the homogeneous region bounded by curves $y = x$ and $y = x^2$.

The curves have intersections in points $[0, 0]$ and $[1, 1]$. See figure.



We write the domain as a normal with respect to the x-axis with inequalities in the form:

$$\Omega: \quad 0 \le x \le 1, \quad x^2 \le y \le x.$$

First we compute the mass of the homogeneous domain

$$m = \iint_{\Omega} \sigma(x,y)\,\mathrm{d}x\mathrm{d}y = \sigma \int_0^1 \mathrm{d}x \int_{x^2}^{x} \mathrm{d}y$$

$$= \sigma \int_0^1 \left(x - x^2\right)\,\mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$

Then we calculate static moments

$$S_x = \iint_{\Omega} y\,\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sigma \int_0^1 \mathrm{d}x \int_{x^2}^{x} y\,\mathrm{d}y = \sigma \int_0^1 \left[\frac{y^2}{2}\right]_{x^2}^{x} \mathrm{d}x\,\mathrm{d}y$$

$$= \sigma \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right)\,\mathrm{d}x = \left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1 = \frac{1}{15}.$$

$$S_y = \iint\limits_{\Omega} x\,\sigma(x,y)\,\mathrm{d}x\,\mathrm{d}y = \sigma \int\limits_0^1 \mathrm{d}x \int\limits_{x^2}^{x} x\,\mathrm{d}y = \sigma \int\limits_0^1 [xy]_{x^2}^{x}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \sigma \int\limits_0^1 \left(x^2 - x^3\right)\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{12}.$$

Coordinates of the center are

$$\xi = \frac{S_y}{m} = \frac{1}{2}, \qquad \eta = \frac{S_x}{m} = \frac{2}{5}$$

and center of mass is

$$C = [\xi, \eta] = \left[\frac{1}{2}, \frac{2}{5}\right].$$

Exercise 23

Compute the center of mass coordinates of the homogeneous region bounded by curves $y = x^2$, $x = 4$, $y = 0$.

2 Triple integral

2.1 Triple integral over rectangular hexahedron

Let $u = f(x,y,z)$ is a function of three variables that is continuous and bounded on the rectangular hexahedron

$$G = \left\{ [x,y,x] \in \mathbb{R}^3 \colon x \in [a,b]\,, y \in [c,d]\,, z \in [e,h] \right\}.$$

We divide intervals $[a,b]\,, [c,d]\,, [e,h]$ by three sequences of points

$$a = x_0 < x_1 < x_2 < \ldots < x_m = b,$$

$$c = y_0 < y_1 < y_2 < \ldots < y_n = d$$

and

$$e = z_0 < z_1 < z_2 < \ldots < z_p = h$$

to intervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, m$,$[y_{j-1}, y_j]$, $j = 1, 2, \ldots, n$ and $[z_{k-1}, z_k]$, $k = 1, 2, \ldots, p$. We denote $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ and $\Delta z_k = z_k - z_{k-1}$.
The planes that lead through points x_i or y_j or z_k parallel to coordinate planes divide the hexahedron G to $m \cdot n \cdot p$ small hexahedrons G_{ijk} (see Figure) with volume of each $\Delta G_{ijk} = \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$. We choose an arbitrary point $[\xi_i, \eta_j, \zeta_k]$ in each hexahedron G_{ijk} and we create products $f(\xi_i, \eta_j, \zeta_k) \cdot \Delta G_{ijk} = f(\xi_i, \eta_j, \zeta_k) \cdot \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$ that for positive function $f(x,y,z) \geq 0$ has a physical meaning of the mass of the hexahedron G_{ijk} with density $f(\xi_i, \eta_j, \zeta_k)$. The sum of these products

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(\xi_i, \eta_j, \zeta_k) \cdot \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

represents the mass of the body consisted of such hexahedrons.

Definition

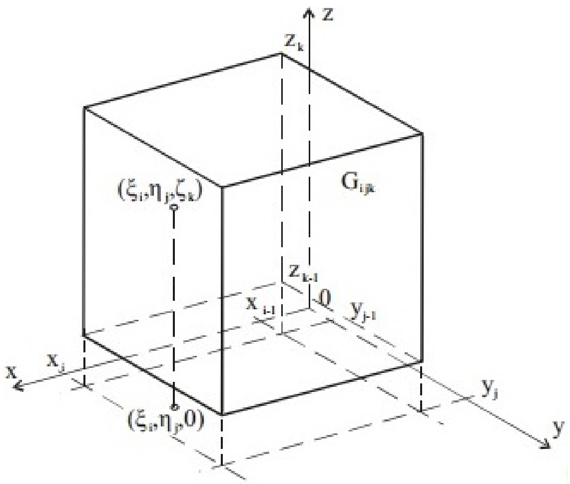
If there exists

$$\lim \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f(\xi_i, \eta_j, \zeta_k) \cdot \Delta x_i \cdot \Delta y_j \cdot \Delta z_k$$

for $m \to \infty$, $n \to \infty$, $p \to \infty$, $\Delta x_i \to 0$, $\Delta y_j \to 0$, $\Delta z_k \to 0$ for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $z = 1, 2, \ldots, k$, we call it a **triple integral** of function $f(x,y,z)$ over the rectangular hexahedron G and denote it by

$$\iiint\limits_G f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

The triple integral over a hexahedron G of a positive function $f(x,y,z) > 0$ has a meaning of the mass of a hexahedron G with density $f(x,y,z)$.



Theorem (Fubini's theorem)

Let $G = \left\{[x,y,z] \in \mathbb{R}^3 \colon x \in [a,b]\,,\, y \in [c,d]\,,\, z \in [e,h]\right\}$. If a function $f(x,y,z)$ is continuous on the hexahedron G, then

$$\iiint\limits_G f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_a^b \left(\int\limits_c^d \left(\int\limits_e^h f(x,y,z)\,\mathrm{d}z\right)\,\mathrm{d}y\right)\,\mathrm{d}x.$$

The Theorem is similar to two-dimensional Fubini's theorem. We can rewrite the formula by using a different order of integration in five more ways. The triple integral is then converted to three one-dimensional integrals. Similarly to the double integral, we can write

$$\int\limits_a^b \left(\int\limits_c^d \left(\int\limits_e^h f(x,y,z)\,\mathrm{d}z\right)\,\mathrm{d}y\right)\,\mathrm{d}x = \int\limits_a^b \mathrm{d}x \int\limits_c^d \mathrm{d}y \int\limits_e^h f(x,y,z)\,\mathrm{d}z.$$

If the integrand $f(x,y,z)$ can be written as a product of three functions of one variable $f(x,y,z) = f_1(x)\cdot f_2(y)\cdot f_3(z)$, it holds:

$$\iiint\limits_G f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_a^b f_1(x)\,\mathrm{d}x \cdot \int\limits_c^d f_2(y)\,\mathrm{d}y \cdot \int\limits_e^h f_3(z)\,\mathrm{d}z.$$

Theorem (Properties of the triple integral over a rectangular hexahedron)

1. $\displaystyle\iiint\limits_G cf(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = c\iiint\limits_G f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

2. $\displaystyle\iiint\limits_G (f(x,y,z)+g(x,y,z))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_G f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$
$\displaystyle+\iiint\limits_G g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

3. $\iiint\limits_G f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{G_1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint\limits_{G_2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$

where f, g are continuous functions on G, $c \in \mathbb{R}$ and G_1, G_2 are non-overlapping hexahedrons that fulfil $G = G_1 \cup G_2$.

Example 24

Compute $\iiint\limits_G xy^2z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over the rectangular hexahedron

$$G = \left\{[x,y,z] \in \mathbb{R}^3 \colon x \in [0,2]\,,\, y \in [1,3]\,,\, z \in [1,2]\right\}.$$

$$\iiint\limits_G xy^2z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_0^2 x\,\mathrm{d}x \cdot \int\limits_1^3 y^2\,\mathrm{d}y \cdot \int\limits_1^2 z\,\mathrm{d}z$$
$$= \left[\frac{x^2}{2}\right]_0^2 \cdot \left[\frac{y^3}{3}\right]_1^3 \cdot \left[\frac{z^2}{2}\right]_1^2 = 2 \cdot \left(9 - \frac{1}{3}\right) \cdot \left(2 - \frac{1}{2}\right)$$
$$= 2 \cdot \frac{26}{3} \cdot \frac{3}{2} = 26$$

Remark

If it is not possible to decompose the integrand as a product of three one-dimensional integrals we can always use Fubini's theorem.

Example 25

Compute $\iiint\limits_G (x+y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over the rectangular hexahedron

$$G = \left\{[x,y,z] \in \mathbb{R}^3 \colon x \in [0,1]\,,\, y \in [0,2]\,,\, z \in [0,3]\right\}.$$

$$\iiint\limits_G (x+y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_0^1 \mathrm{d}x \int\limits_0^2 \mathrm{d}y \int\limits_0^3 (x+y)\,\mathrm{d}z = \int\limits_0^1 \mathrm{d}x \int\limits_0^2 (x+y)\,[z]_0^3\,\mathrm{d}y$$
$$= 3\int\limits_0^1 \mathrm{d}x \int\limits_0^2 (x+y)\,\mathrm{d}y = 3\int\limits_0^1 \left[xy + \frac{y^2}{2}\right]_0^2 \mathrm{d}x = 3\int\limits_0^1 (2x+2)\,\mathrm{d}x$$
$$= 6\left[\frac{x^2}{2} + x\right]_0^1 = 6 \cdot \frac{3}{2} = 9$$

Exercise 26

Compute following integrals over their domains G.

a) $\iiint\limits_G xy^2\sqrt{z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad G = \left\{[x,y,z] \in \mathbb{R}^3 \colon x \in [-2,1]\,,\, y \in [1,3]\,,\, z \in [2,4]\right\}$

b) $\displaystyle\iiint_G \frac{1}{1-x-y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad G=\left\{[x,y,z]\in\mathbb{R}^3\colon x\in[0,1]\,,\, y\in[2,5]\,,\, z\in[2,4]\right\}$

c) $\displaystyle\iiint_G \ln x^{yz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad G=\left\{[x,y,z]\in\mathbb{R}^3\colon x\in[1,2]\,,\, y\in[0,1]\,,\, z\in[0,2]\right\}$

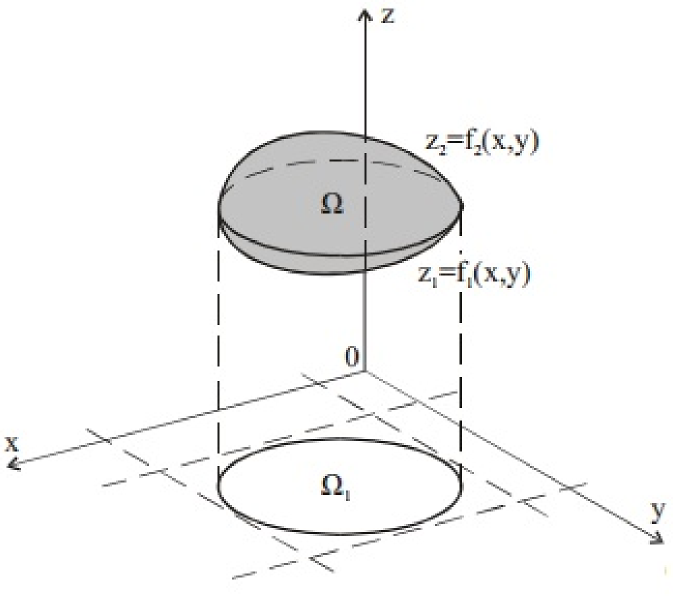
d) $\displaystyle\iiint_G \left(\frac{1}{x}+\frac{2}{y}+\frac{3}{z}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad G=\left\{[x,y,z]\in\mathbb{R}^3\colon x\in[1,2]\,,\, y\in[1,2]\,,\, z\in[1,2]\right\}$

e) $\displaystyle\iiint_G e^{x+y+z}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad G=\left\{[x,y,z]\in\mathbb{R}^3\colon x\in[0,1]\,,\, y\in[0,1]\,,\, z\in[0,1]\right\}$

f) $\displaystyle\iiint_G \sqrt{xyz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad G=\left\{[x,y,z]\in\mathbb{R}^3\colon x\in[0,1]\,,\, y\in[0,9]\,,\, z\in[0,16]\right\}$

2.2 Triple integral over a general domain

Similarly like in two-dimensional case, we are able to generalize our problem of solving triple integrals over any three-dimensional region Ω that is bounded by a closed surface. We consider only such surfaces that don't intersect themselves and lines parallel with z-axis leading through an arbitrary inner point of the surface intersect with the surface in two points. Such domain will be called **normal domain with respect to the coordinate plane** xy.



We create an orthogonal projection Ω_1 of the domain Ω into xy-plane. A variable z must fulfil

$$f_1(x,y)\le z\le f_2(x,y).$$

The domain Ω_1 is either normal with respect to x-axis or y-axis and we describe it using approach from the Double integral section by inequalities $x_1\le x\le x_2$, $g_1(x)\le y\le g_2(x)$ resp. $y_1\le y\le y_2$, $h_1(y)\le x\le h_2(y)$. If the function $f(x,y,z)$ is continuous on Ω we use a method similar to Fubini's theorem used in the Double integral section and express

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_{x_1}^{x_2} \mathrm{d}x \int\limits_{g_1(x)}^{g_2(x)} \mathrm{d}y \int\limits_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,\mathrm{d}z$$

resp.

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_{y_1}^{y_2} \mathrm{d}y \int\limits_{h_1(y)}^{h_2(y)} \mathrm{d}x \int\limits_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,\mathrm{d}z.$$

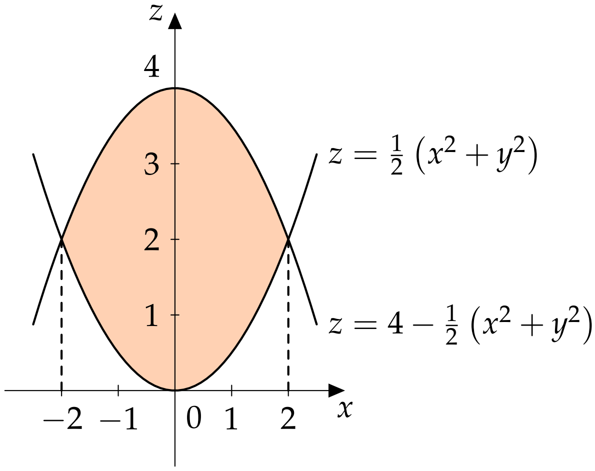
We start to integrate with respect to variable z, limits are functions of two variables x, y. After that we calculate a double integral over a regular domain Ω_1.
We are able to use analogical approach and create an orthogonal projection of the domain Ω into planes either xz or yz. That way we can use six different orders of integration for an arbitrary domain.

Remark

The triple integral over a general closed domain has analogical properties as the triple integral over a rectangular hexahedron.

Example 27

Determine integration limits for integral $\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over the domain Ω that is bounded by surfaces $z = \frac{1}{2}\left(x^2+y^2\right)$ and $z = 4 - \frac{1}{2}\left(x^2+y^2\right)$.



Both surfaces are rotational paraboloids and based on the figure where we can see projection of the body into yz-plane

$$\frac{1}{2}\left(x^2+y^2\right) \le z \le 4 - \frac{1}{2}\left(x^2+y^2\right).$$

We obtain an equation of the intersection of both surfaces from the equation

$$\frac{1}{2}\left(x^2+y^2\right) = 4 - \frac{1}{2}\left(x^2+y^2\right)$$

which leads to

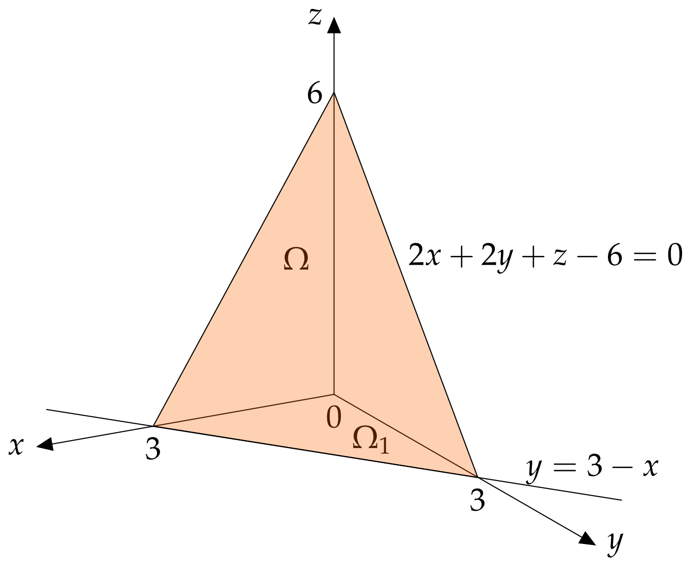
$$x^2 + y^2 = 4.$$

Therefore, the orthogonal projection Ω_1 of the domain Ω to coordinate plane xy is a circle with the center in the origin of coordinates and radius $r = 2$. That domain can be treated as a normal domain with respect to the x-axis with inequalities

$$-2 \le x \le 2,$$
$$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}.$$

Example 28

Compute integral $\iiint\limits_{\Omega} x \, dx \, dy \, dz$ over the domain Ω bounded by surfaces $x = 0$, $y = 0$, $z = 0$, $2x + 2y + z - 6 = 0$.



Based on the figure, the domain Ω must fulfil

$$0 \le z \le 6 - 2x - 2y$$

The orthogonal projection Ω_1 of the domain Ω to xy-plane is the triangle bounded by lines $x = 0$, $y = 0$, $y = 3 - x$. The last equation is intersection of planes $2x + 2y + z - 6 = 0$ and $z = 0$.
We describe Ω_1 as normal with respect to x-axis by inequalities

$$\Omega_1: \quad \begin{aligned} &0 \le x \le 3, \\ &0 \le y \le 3 - x \end{aligned}$$

and we can calculate the integral

$$\iiint\limits_{\Omega} x \, dx \, dy \, dz = \int_0^3 dx \int_0^{3-x} dy \int_0^{6-2x-2y} x \, dz = \int_0^3 dx \int_0^{3-x} [xz]_0^{6-2x-2y} \, dy$$

$$= \int_0^3 dx \int_0^{3-x} x(6 - 2x - 2y) \, dy = \int_0^3 x \left[6y - 2xy - y^2\right]_0^{3-x} dx$$

$$= \int_0^3 x\left[6(3-x)-2x(3-x)-(3-x)^2\right]\,\mathrm{d}x = \int_0^3 (x^3-6x^2+9x)\,\mathrm{d}x$$

$$= \left[\frac{x^4}{4}-2x^3+\frac{9}{2}x^2\right]_0^3 = \frac{27}{4}.$$

Exercise 29

Compute following integrals over their domains Ω.

a) $\displaystyle\iiint_\Omega x^3y^2z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \{[x,y,z]\colon y\geq 0,\ y\leq x,\ x\leq 1,\ z\geq 0,\ z\leq xy\}$

b) $\displaystyle\iiint_\Omega \frac{1}{1+x+y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \{[x,y,z]\colon x\geq 0,\ y\geq 0,\ z\geq 0,\ x+y+z\leq 1\}$

c) $\displaystyle\iiint_\Omega \frac{x+z}{4+y}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \{[x,y,z]\colon x\geq 0,\ y\geq 0,\ z\geq 0,\ x+z\leq 3,\ y\leq 4\}$

d) $\displaystyle\iiint_\Omega y\cos(x+z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z]\colon y\leq\sqrt{x},\ y\geq 0,\ z\geq 0,\ x+z\leq\frac{\pi}{2}\right\}$

e) $\displaystyle\iiint_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z]\colon x+y\leq 1,\ y\geq 0,\ y\leq 2x,\ z\geq 0,\ z\leq 1-x^2\right\}$

f) $\displaystyle\iiint_\Omega z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z]\colon x^2+y^2\leq 4,\ 0\leq z\leq 2,\ y\geq 0\right\}$

2.3 Transformation of the triple integral

Similarly to the double integral, using Cartesian coordinates for some domains can be rather complicated. Especially in case of cylinders, cones or spheres. Therefore, we formulate analogical theorem that describes general transformation of the triple integral.

Theorem (Transformation to general coordinates)

- Let equations $x = u(r,s,t), y = v(r,s,t), z = w(r,s,t)$ map the region Ω bijectively to the region Ω^*.
- Let function $f(x,y,z)$ be continuous and bounded on Ω and functions $x = u(r,s,t)$, $y = v(r,s,t)$, $z = w(r,s,t)$ have continuous partial derivatives on $\hat{\Omega}$ that fulfils $\Omega^* \subset \hat{\Omega}$.
- Let $J(u,v,w) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{vmatrix} \neq 0$ in Ω^*.

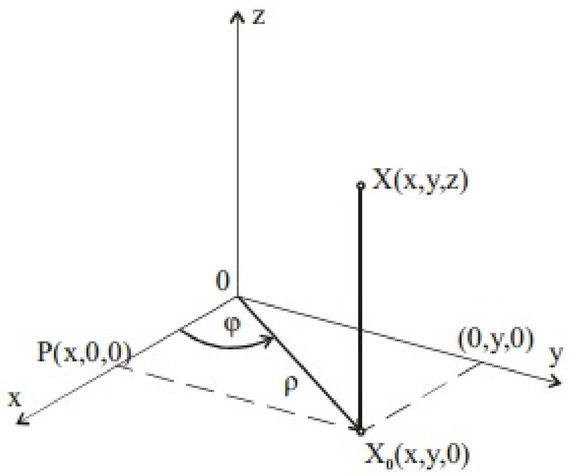
Then

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{\Omega^*} f(u(r,s,t),v(r,s,t),w(r,s,t))|J|\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t.$$

Determinant $J(u,v,w) = \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \\ \frac{\partial w}{\partial r} & \frac{\partial w}{\partial s} & \frac{\partial w}{\partial t} \end{vmatrix}$ is again called **Jacobian** or **Jacobi determinant**.

2.3.1 Transformation to cylindrical coordinates

Transformation to cylindrical coordinates is suitable for integration domains such as cylinders, cones or their parts. It is used in cases when orthogonal projection Ω_1 of the domain Ω to plane xy is a disc or a part of a disc. We replace Cartesian coordinates x, y, z by cylindrical coordinates ρ, φ, z, according to the following figure.



The meaning of coordinates ρ, φ is the same as we have already used for polar coordinates and the third coordinate z doesn't change.

The transformation to cylindrical coordinates is given by transformation equations

$$\begin{aligned} x &= \rho\cos\varphi, \\ y &= \rho\sin\varphi, \\ z &= z. \end{aligned}$$

According to theorem describing transformation to general coordinates, we replace volume element $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ by $|J|\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z$, where the Jacobian of the transformation to cylindrical coordinates satisfies

$$J(\rho,\varphi,z) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho.$$

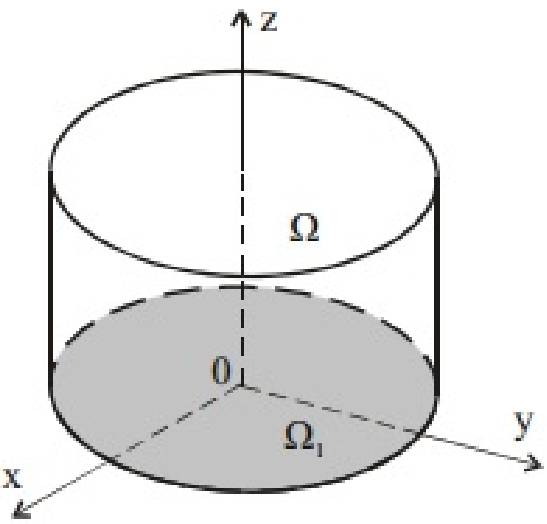
The transformation of the triple integral to cylindrical coordinates can then be written in the form

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{\Omega^*} f(\rho\cos\varphi,\rho\sin\varphi,z)\rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z.$$

Example 30

Compute $\iiint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over the domain Ω bounded by surfaces $x^2+y^2=1$, $z=0$, $z=1$.

The domain Ω is the rotational cylinder symetrical with respect to the z-axis, with radius of the base $\rho=1$ and height $z=1$, according to the following figure.



We need to determine the bounds of the transformed domain Ω^*. It is obvious that $0 \le z \le 1$. Inequalities for coordinates ρ,φ are the same as for transformation to polar coordinates, i.e. $0 \le \rho \le 1$, $0 \le \varphi \le 2\pi$. Therefore

$$\iiint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{\Omega^*} \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\varphi \cdot \int_0^1 \rho\,\mathrm{d}\rho \cdot \int_0^1 \mathrm{d}z$$

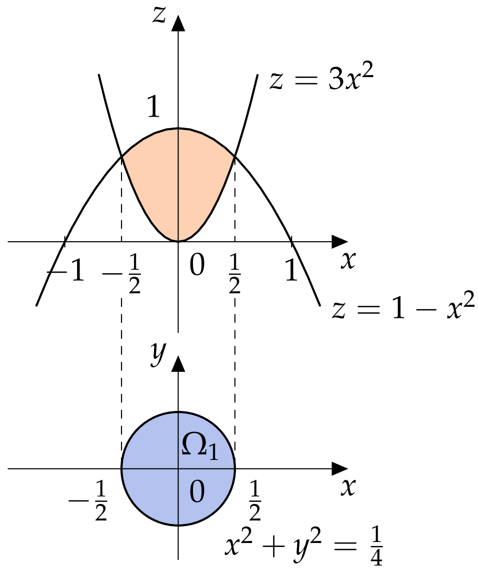
$$= [\varphi]_0^{2\pi} \cdot \left[\frac{\rho^2}{2}\right]_0^1 \cdot [z]_0^1 = \pi.$$

Example 31

Compute $\iiint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ over the domain Ω bounded by surfaces $z = 3x^2 + 3y^2$, $z = 1 - x^2 - y^2$.

Both surfaces are paraboloids and the orthogonal projection Ω_1 of the domain Ω to coordinate plane x, y is a ring, whose equation we obtain from the intersection of both paraboloids

$$\begin{aligned} 3x^2 + 3y^2 &= 1 - x^2 - y^2 \\ x^2 + y^2 &= \frac{1}{4}. \end{aligned}$$



Therefore, inequalities for coordinates ρ, φ must fulfil

$$0 \le \rho \le \frac{1}{2}, \qquad 0 \le \varphi \le 2\pi.$$

We obtain limits of the variable z from equations of both paraboloids by substituting of variables

$$\begin{aligned} z = 3x^2 + 3y^2 = 3\rho^2 \cos^2\varphi + 3\rho^2 \sin^2\varphi = 3\rho^2, \\ z = 1 - x^2 - y^2 = 1 - \rho^2\cos^2\varphi - \rho^2 \sin^2\varphi = 1 - \rho^2. \end{aligned}$$

Hence

$$3\rho^2 \le z \le 1 - \rho^2.$$

We compute the integral by using transformation to cylindrical coordinates

$$\iiint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint\limits_{\Omega^*} \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{1/2} \mathrm{d}\rho \int_{3\rho^2}^{1-\rho^2} \rho\,\mathrm{d}z$$

$$= 2\pi \int_0^{1/2} \rho[z]_{3\rho^2}^{1-\rho^2}\,\mathrm{d}\rho = 2\pi \int_0^{1/2} \rho\left(1 - 4\rho^2\right)\,\mathrm{d}\rho = 2\pi \int_0^{1/2} \left(\rho - 4\rho^3\right)\,\mathrm{d}\rho$$

$$= 2\pi \left[\frac{\rho^2}{2} - \rho^4\right]_0^{1/2} = 2\pi \cdot \frac{1}{16} = \frac{\pi}{8}.$$

Exercise 32

Compute following integrals over their domains Ω.

a) $\displaystyle\iiint_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z] \in \mathbb{R}^3 \colon x^2+y^2 \le 1,\, x \ge 0,\, 0 \le z \le 6\right\}$

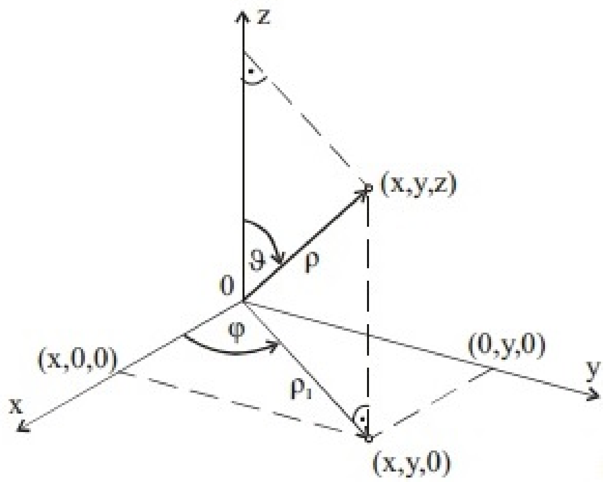
b) $\displaystyle\iiint_\Omega z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z] \in \mathbb{R}^3 \colon x^2+y^2 \le 9,\, x \le y \le x\sqrt{3},\, 0 \le z \le 4\right\}$

c) $\displaystyle\iiint_\Omega z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z] \in \mathbb{R}^3 \colon z \ge \sqrt{x^2+y^2},\, z \le 1\right\}$

d) $\displaystyle\iiint_\Omega z\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \quad \Omega = \left\{[x,y,z] \in \mathbb{R}^3 \colon x^2+y^2 \le 2x,\, 0 \le z \le 1\right\}$

2.3.2 Transformation to spherical coordinates

Transformation to spherical coordinates is suitable for integrals where integration domains are spheres, ellipsoids or their parts. We replace Cartesian coordinates x, y, z by spherical coordinates ρ, φ, ϑ according to the following figure.



The coordinate ρ denotes a distance between the point $[x,y,z]$ and the origin of the coordinates, φ denotes positively oriented angle in coordinate xy-plane between positive part of the x-axis and the projection ρ_1 of the radius vector ρ to coordinate xy-plane and ϑ denotes positively oriented angle between positive part of the z-axis and the radius vector ρ.

We obtain transformation equations

$$
\begin{aligned}
x &= \rho\cos\varphi\sin\vartheta,\\
y &= \rho\sin\varphi\sin\vartheta,\\
z &= \rho\cos\vartheta.
\end{aligned}
$$

Jacobian of the transformation to spherical coordinates satisfies

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial \vartheta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial \vartheta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial \vartheta} \end{vmatrix} = \begin{vmatrix} \cos\varphi\sin\vartheta & -\rho\sin\varphi\sin\vartheta & \rho\cos\varphi\cos\vartheta \\ \sin\varphi\sin\vartheta & \rho\cos\varphi\sin\vartheta & \rho\sin\varphi\cos\vartheta \\ \cos\vartheta & 0 & -\rho\sin\vartheta \end{vmatrix}$$

$$= -\rho^2 \sin\vartheta.$$

Transformation of the triple integral to spherical coordinates can be written in the form

$$\iiint\limits_{\Omega} f(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$= \iiint\limits_{\Omega^*} f(\rho\cos\varphi\sin\vartheta, \rho\sin\varphi\sin\vartheta, \rho\cos\vartheta)\rho^2 \sin\vartheta\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\vartheta.$$

The sphere with the center in the origin of coordinates and radius a transformed to spherical coordinates is mapped to domain Ω^* given by inequalities

$$\Omega^*: \quad \begin{aligned} 0 &\le \rho \le a, \\ 0 &\le \varphi \le 2\pi, \\ 0 &\le \vartheta \le \pi. \end{aligned}$$

Example 33

Compute $\iiint\limits_{\Omega} \left(x^2+y^2+z^2\right)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ over the domain Ω bounded by $x^2+y^2+z^2 \ge 1$ and $x^2+y^2+z^2 \le 4$.

The domain Ω is bounded by two spherical surfaces with the center in the origin of coordinates and radii $\rho_1 = 1$, $\rho_2 = 2$. We use transformation to spherical coordinates with inequalities

$$\Omega^*: \quad \begin{aligned} 1 &\le \rho \le 2, \\ 0 &\le \varphi \le 2\pi, \\ 0 &\le \vartheta \le \pi. \end{aligned}$$

and calculate the integral using transformation to spherical coordinates

$$\iiint\limits_{\Omega} \left(x^2+y^2+z^2\right)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$= \iiint\limits_{\Omega^*} \left(\rho^2\cos^2\varphi\sin^2\vartheta + \rho^2\sin^2\varphi\sin^2\vartheta + \rho^2\cos^2\vartheta\right)\rho^2\sin\vartheta\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\vartheta$$

$$= \int_1^2 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi} \rho^4 \sin\vartheta \,\mathrm{d}\vartheta = \int_1^2 \rho^4 \,\mathrm{d}\rho \cdot \int_0^{2\pi} \mathrm{d}\varphi \cdot \int_0^{\pi} \sin\vartheta \,\mathrm{d}\vartheta$$

$$= \left[\frac{\rho^5}{5}\right]_1^2 \cdot [\varphi]_0^{2\pi} \cdot [-\cos\vartheta]_0^{\pi} = \frac{31}{5} \cdot 2\pi \cdot 2 = \frac{124}{5}\pi.$$

Example 34

Compute integral $\iiint\limits_{\Omega} z \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$ over the domain Ω bounded by $x^2 + y^2 + z^2 \leq 4$, $x \geq 0, y \geq 0, z \geq 0$.

The domain Ω is one eighth of the sphere in the first octant with the center in the origin of coordinates and radius $\rho = 2$. Therefore

$$\Omega^*: \quad \begin{aligned} &0 \leq \rho \leq 2, \\ &0 \leq \varphi \leq \frac{\pi}{2}, \\ &0 \leq \vartheta \leq \frac{\pi}{2}. \end{aligned}$$

and

$$\iiint\limits_{\Omega} z \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z = \iiint\limits_{\Omega^*} \rho \cos\vartheta \,\rho^2 \sin\vartheta \,\mathrm{d}\rho \,\mathrm{d}\varphi \,\mathrm{d}\vartheta$$

$$= \int_0^2 \mathrm{d}\rho \int_0^{\pi/2} \mathrm{d}\varphi \int_0^{\pi/2} \rho^3 \sin\vartheta \cos\vartheta \,\mathrm{d}\vartheta = \int_0^2 \rho^3 \,\mathrm{d}\rho \cdot \int_0^{\pi/2} \mathrm{d}\varphi \cdot \int_0^{\pi/2} \frac{1}{2} \sin 2\vartheta \,\mathrm{d}\vartheta$$

$$= \left[\frac{\rho^4}{4}\right]_0^2 \cdot [\varphi]_0^{\pi/2} \cdot \left[-\frac{1}{4}\cos 2\vartheta\right]_0^{\pi/2} = 4 \cdot \frac{\pi}{2} \cdot \frac{1}{2} = \pi.$$

Exercise 35

Compute following integrals over their domains Ω.

a) $\iiint\limits_{\Omega} z(x^2 + y^2) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z, \quad \Omega = \left\{[x,y,z] : x^2 + y^2 + z^2 \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\right\}$

b) $\iiint\limits_{\Omega} \sqrt{x^2 + y^2 + z^2} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z,$

$\Omega = \left\{[x,y,z] : x^2 + y^2 + z^2 \leq 1,\ x \geq 0,\ y \geq 0,\ z \geq 0\right\}$

c) $\iiint\limits_{\Omega} (x^2 + y^2) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z, \quad \Omega = \left\{[x,y,z] : 4 \leq x^2 + y^2 + z^2 \leq 9,\ z \geq 0\right\}$

d) $\iiint\limits_{\Omega} \frac{\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z}{1 + x^2 + y^2 + z^2}, \quad \Omega = \left\{[x,y,z] : x^2 + y^2 + z^2 \leq 4,\ x \leq y \leq x\sqrt{3},\ z \geq 0\right\}$

2.4 Practical applications of the triple integral

2.4.1 Volume of a body

The volume of the body Ω is given by

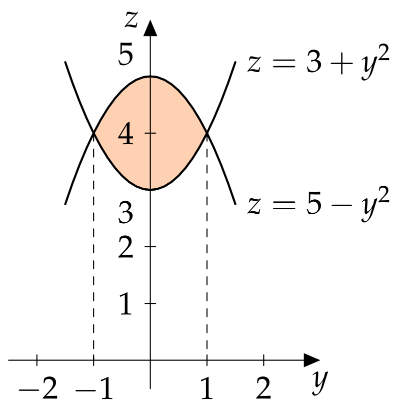
$$V = \iiint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Example 36

Compute the volume of the body bounded by cylindrical surfaces $z = 5 - y^2$, $z = y^2 + 3$ and planes $x = 0$, $x = 2$.

While limits of variables x, z are obvious, we need to calculate intersections of cylindrical surfaces to determine the limits of variable y.

$$\begin{aligned} 5 - y^2 &= y^2 + 3 \\ 2y^2 &= 2 \\ y &= \pm 1. \end{aligned}$$



Therefore, inequalities for the domain Ω are in the form

$$\Omega: \quad \begin{aligned} 0 \le x &\le 2, \\ -1 \le y &\le 1, \\ y^2 + 3 \le z &\le 5 - y^2. \end{aligned}$$

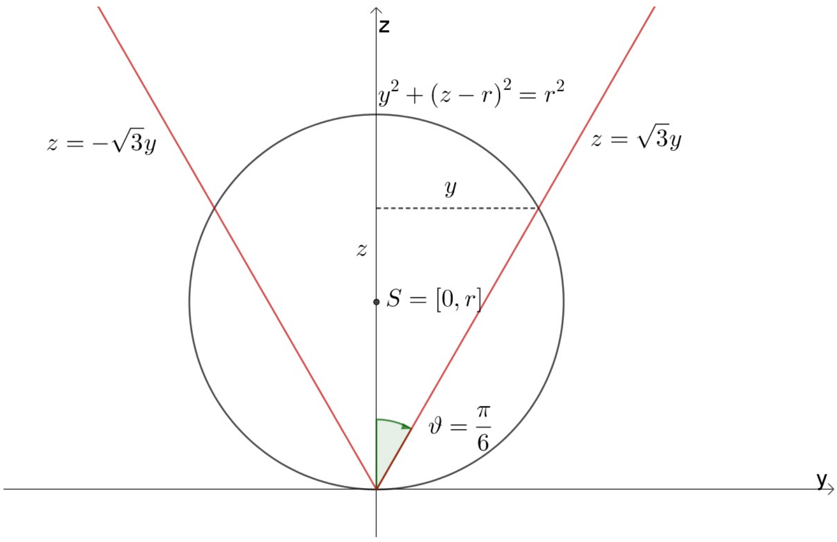
We calculate volume of the body by using the triple integral

$$V = \iiint\limits_{\Omega} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\limits_0^2 \mathrm{d}x \int\limits_{-1}^1 \mathrm{d}y \int\limits_{y^2+3}^{5-y^2} \mathrm{d}z = \int\limits_0^2 \mathrm{d}x \int\limits_{-1}^1 [z]_{y^2+3}^{5-y^2}\,\mathrm{d}y$$

$$= \int\limits_0^2 \mathrm{d}x \cdot \int\limits_{-1}^1 (2 - 2y^2)\;\mathrm{d}y = 2\left[2y - \frac{2y^3}{3}\right]_{-1}^1 = 2 \cdot \frac{8}{3} = \frac{16}{3}.$$

Example 37

Compute the volume of the body bounded by surfaces $x^2 + y^2 + (z-r)^2 = r^2$ and $z = \sqrt{3x^2 + 3y^2}$.



The equation $x^2 + y^2 + (z-r)^2 = r^2$ describes the sphere with radius r and the center in point $S = [0, 0, r]$. The second equation $z = \sqrt{3x^2 + 3y^2}$ describes the cone oriented along the z-axis with vertex in the coordinates origin. We will transform the problem to spherical coordinates. While the limits for azimuth φ must be $0 \le \varphi \le 2\pi$, we have to find also the limits for angle ϑ. Let us make the projection of the domain to yz-plane, which is visible on the figure. By putting $x = 0$ in the equation of the cone $z = \sqrt{3x^2 + 3y^2}$ we obtain the equation of both lines

$$z = \sqrt{3y^2} = \sqrt{3}|y|.$$

Therefore, the upper limit for the angle ϑ must fulfil

$$\tan\vartheta = \frac{y}{z} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Hence, the upper limit for $\vartheta = \frac{\pi}{6}$. We can also find the upper limit for radius ρ. We transform the equation of the sphere to $x^2 + y^2 + z^2 - 2zr = 0$ and then to spherical coordinates

$$\begin{aligned} \rho^2 - 2r\rho\cos\vartheta &= 0 \\ \rho(\rho - 2r\cos\vartheta) &= 0 \end{aligned}$$

We have

$$\begin{aligned} \Omega^*: \quad & 0 \le \varphi \le 2\pi, \\ & 0 \le \vartheta \le \frac{\pi}{6}, \\ & 0 \le \rho \le 2r\cos\vartheta. \end{aligned}$$

The limits of variable ρ depends on variable ϑ, therefore we have to start the calculation with inner integral with respect to variable ρ.

$$V = \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi/6} \mathrm{d}\vartheta \int_0^{2r\cos\vartheta} \rho^2 \sin\vartheta \,\mathrm{d}\rho = 2\pi \int_0^{\pi/6} \sin\vartheta \left[\frac{\rho^3}{3}\right]_0^{2r\cos\vartheta} \mathrm{d}\vartheta$$

$$= \frac{16}{3}\pi r^3 \int_0^{\pi/6} \sin\vartheta \cos^3\vartheta \,\mathrm{d}\vartheta = \left| \begin{array}{ll} t = \cos\vartheta & 0 \to 1 \\ \mathrm{d}t = -\sin\vartheta\,\mathrm{d}\vartheta & \frac{\pi}{6} \to \frac{\sqrt{3}}{2} \end{array} \right|$$

$$= -\frac{16}{3}\pi r^3 \int_1^{\sqrt{3}/2} t^3 \,\mathrm{d}t = \frac{16}{3}\pi r^3 \left[\frac{t^4}{4}\right]_{\sqrt{3}/2}^1 = \frac{7}{12}\pi r^3.$$

Exercise 38

Compute the volume of body bounded by surfaces.

a) $z = x^2 + y^2$, $z = y$

b) $x - y + z = 6$, $x + y = 2$, $x = y$, $y = 0$, $z = 0$

c) $y = x^2$, $z = 0$, $y + z = 2$

d) $y = \ln x$, $y = \ln^2 x$, $z = 0$, $y + z = 1$

2.4.2 Mass of a body

Mass of the body Ω is given by

$$m = \iiint_\Omega \sigma(x, y, z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$$

where $\sigma(x, y, z) > 0$ denotes volume density in each point of the domain Ω.

Example 39

Compute the mass of the body bounded by surfaces $x^2 + y^2 + z^2 \leq 4$. The volume density in each point of Ω is equal to its distance to the coordinates origin.

The domain Ω is the sphere with center in the coordinate origin and radius 2. Therefore we will calculate the problem by using transformation to spherical coordinates with transformation equations

$$\begin{aligned} x &= \rho\cos\varphi\sin\vartheta, \\ y &= \rho\sin\varphi\sin\vartheta, \\ z &= \rho\cos\vartheta. \end{aligned}$$

The volume density in each point of Ω is equal to its distance to the coordinates origin, therefore

$$\sigma = \sqrt{x^2 + y^2 + z^2} = \rho.$$

After the transformation to spherical coordinates, the domain is rectangular given by inequations

$$\Omega^* \quad \begin{aligned} 0 \le \rho \le 2, \\ 0 \le \varphi \le 2\pi, \\ 0 \le \vartheta \le \pi. \end{aligned}$$

Now, we can calculate mass of the sphere

$$m = \iiint\limits_{\Omega} \sqrt{x^2+y^2+z^2}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint\limits_{\Omega^*} \rho \cdot \rho^2 \sin\vartheta\, \mathrm{d}\rho\, \mathrm{d}\varphi\, \mathrm{d}\vartheta$$

$$= \int_0^{2\pi} \mathrm{d}\varphi \cdot \int_0^{\pi} \sin\vartheta\, \mathrm{d}\vartheta \cdot \int_0^{2} \rho^3\, \mathrm{d}\rho = [\varphi]_0^{2\pi} \cdot [-cos\vartheta]_0^{\pi} \cdot \left[\frac{\rho^4}{4}\right]_0^2$$

$$= 2\pi \cdot 2 \cdot 4 = 16\pi.$$

Exercise 40

a) Compute the mass of body $x^2+y^2+z^2 \le 1$ with density $\sigma = \dfrac{2}{x^2+y^2+z^2}$.

b) Compute the mass of body bounded by surfaces $x^2 = 2y$, $y+z=1$, $2y+z=2$, with density $\sigma = y$.

2.4.3 Statical moments and moments of inertia

Let the domain Ω is a body with given density $\sigma(x,y,z) > 0$ in each point $[x,y,z] \in \Omega$. **Statical moment of a body** S_{xy} or S_{xz} or S_{xz} to coordinate plane xy or xz or yz is defined by

$$S_{xy} = \iiint\limits_{\Omega} z\, \sigma(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

$$S_{xz} = \iiint\limits_{\Omega} y\, \sigma(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

$$S_{yz} = \iiint\limits_{\Omega} x\, \sigma(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

The coordinations of **center of mass** $C = [\xi, \eta, \zeta]$ can then by calculated by

$$\xi = \frac{S_{yz}}{m}, \qquad \eta = \frac{S_{xz}}{m}, \qquad \zeta = \frac{S_{xy}}{m},$$

where m is the mass of the body.
Moment of inertia of the body rotating around the x-axis resp. y-axis resp. z-axis is given by

$$I_x = \iiint\limits_{\Omega} \left(y^2+z^2\right) \sigma(x,y,z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z,$$

$$I_y = \iiint\limits_{\Omega} \left(x^2 + z^2\right) \sigma(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

$$I_z = \iiint\limits_{\Omega} \left(x^2 + y^2\right) \sigma(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Exercise 41

a) Calculate the statical moments of a body $x^2 + y^2 + z^2 \leq 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$ to xy-plane. Consider constant density σ.

b) Calculate the statical moments of a cone with radius of the base $r = 3$ and height $h = 2$ to plane that is parallel to the base going through the vertex of the cone. Consider constant density σ.

c) Calculate the moments of inertia of the body bounded by surfaces $x + 2y + 3z = 1$, $x = 0$, $y = 0$, $z = 0$ rotating around y-axis. Consider constant density σ.

d) Calculate the moments of inertia of the body bounded by surfaces $x^2 + y^2 = z^2$, $z = 1$ rotating around z-axis. Consider constant density σ.

3 Theory of the field

3.1 Vector function

Definition

Let $D \subseteq \mathbb{R}$. A **vector function** of one real variable $t \in D$ is defined as a function of one real variable whose range is a vector

$$\mathbf{f}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k} = (x(t), y(t), z(t))$$

Components $x(t), y(t), z(t)$ are real functions of variable t.

From the geometrical point of view the vector function $\mathbf{f}(t)$ describes the set of points in three-dimensional space with coordinates $[x(t), y(t), z(t)]$, $t \in D$. It will create the **graph of the vector function**.
If $x(t), y(t), z(t)$ are continuous for each $t \in D = [a, b]$, then continuous vector function $\mathbf{f}(t)$ defines three-dimensional curve, whose parametrical equations are given by $x = x(t)$, $y = y(t)$, $z = z(t)$, $t \in [a, b]$. From the physical point of view the vector function represent the trajectory of moving mass point.
We can define all key concepts of calculus also for vector functions - limits, continuity, derivatives, indefinite and definite integral. The calculation is made for each component separately. We can also use all concepts of vector algebra for vector functions - operations with vectors, inner and vector product.

Example 42

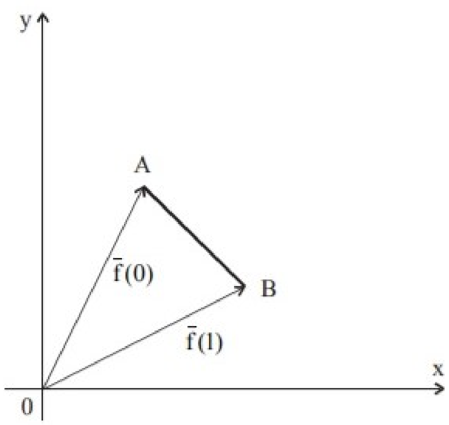
Draw the graph of the vector function

$$\mathbf{f} = (1 + t)\,\mathbf{i} + (2 - t)\,\mathbf{j}, \quad t \in [0, 1].$$

The function is continuous on its domain. The graph is a two-dimensional curve. Parametrical equations of curve

$$\begin{aligned} x &= 1 + t, \\ y &= 2 - t, \qquad t \in [0, 1] \end{aligned}$$

describes the segment of line AB, given by $A = [x(0), y(0)] = [1, 2]$ and
$B = [x(1), y(1)] = [2, 1]$, see figure.



Example 43

Draw the graph of the vector function

$$\mathbf{f} = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}, \quad t \in [0, 2\pi].$$

Function is continuous on its domain. The graph is a two-dimensional curve. Parametrical equations of curve

$$\begin{aligned} x &= 3\cos t, \\ y &= 3\sin t, \qquad t \in [0, 2\pi] \end{aligned}$$

describes circle with center in the coordinate origin and radius $r = 3$, see figure. Starting and ending point of the curve is the same

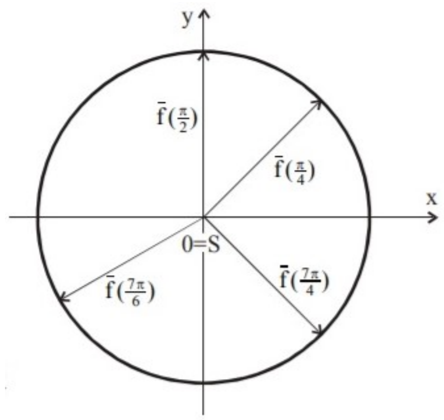
$$A = B = [x(0), y(0)] = [x(2\pi), y(2\pi)] = [3, 0].$$

We can prove it by raising both parametrical equations to the second power

$$x^2 = 9\cos^2 t, \qquad y^2 = 9\sin^2 t$$

and summing them together

$$x^2 + y^2 = 9(\cos^2 t + \sin^2 t) = 9.$$



Example 44

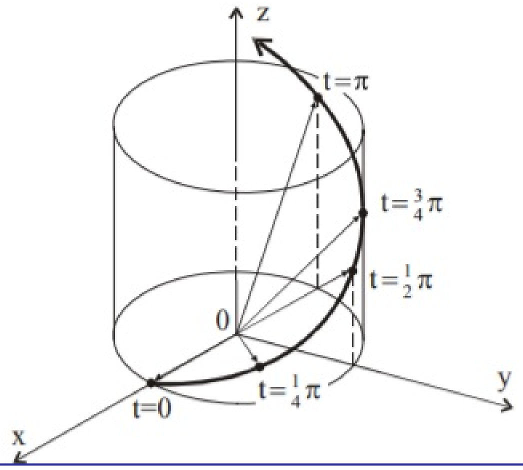
Draw the graph of the vector function

$$\mathbf{f} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}, \quad t \in [0, +\infty).$$

Function is continuous on its domain. The graph is a three-dimensional curve. Parametrical equations of curve

$$\begin{aligned} x &= \cos t, \\ y &= \sin t, \\ z &= t, \qquad t \in [0, \infty) \end{aligned}$$

define the screw line with starting point [1,0,0] on cylindrical surface $x^2 + y^2 = 1$. Analogically to the previous example, we can obtain this equation by raising first two parametrical equations to the second power and summing them together, see figure.



Exercise 45

Draw the graph of the vector function.

a) $\mathbf{f} = 2\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}, \qquad t \in [0, 2\pi)$

b) $\mathbf{f} = t^2\,\mathbf{i} + t\,\mathbf{j}, \qquad t \in (-\infty, +\infty)$

3.2 Scalar field

Definition

Scalar field on the domain $\Omega \subset \mathbb{R}^3$ is given by an scalar function $u = u(x, y, z)$ defined on Ω.

Scalar field assigns one real number (scalar) to each point in Ω. The rate of change of the scalar field is given by directional derivative.

Definition

Let the scalar field $u = u(x, y, z)$ is given on the domain Ω, point $A = [a_1, a_2, a_3] \in \Omega$ and unit vector $\mathbf{s} = (s_1, s_2, s_3)$. We define the limit

$$\lim_{t \to 0^+} \frac{u(A + t\mathbf{s}) - u(A)}{t}$$

as **directional derivative** of the scalar field $u(x, y, z)$ in the point A along the vector $\mathbf{s}$ and we denote it by $\dfrac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}}$.

Theorem

Let partial derivatives of the scalar function u exist in the point $A \in \Omega$. The directional derivative of the scalar field $u(x, y, z)$ in the point A along the unit vector $\mathbf{s}$ can be written in the form

$$\frac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}} = \frac{\partial u(A)}{\partial x} s_1 + \frac{\partial u(A)}{\partial y} s_2 + \frac{\partial u(A)}{\partial z} s_3.$$

The directional derivative of the scalar field $u(x,y,z)$ in the point A along the vector $\mathbf{s}$ determines the slope of the scalar field $u(x,y,z)$ in the point A along the vector $\mathbf{s}$, i.e. rate of change of the scalar field $u(x,y,z)$ in the point A in the direction of the vector $\mathbf{s}$.

Definition

The vector function

$$\mathbf{grad}\, u = \frac{\partial u(x,y,z)}{\partial x}\mathbf{i} + \frac{\partial u(x,y,z)}{\partial y}\mathbf{j} + \frac{\partial u(x,y,z)}{\partial z}\mathbf{k} = \left(\frac{\partial u(x,y,z)}{\partial x}, \frac{\partial u(x,y,z)}{\partial y}, \frac{\partial u(x,y,z)}{\partial z}\right)$$

is called the **gradient of the scalar field** $u(x,y,z)$.

The direction of the greatest increase of the scalar field is given by gradient of the scalar field.

- By implementation of the **Hamilton operator (nabla operator)**

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

  we can write the gradient of the scalar field $u(x,y,z)$ in the form

$$\mathbf{grad}\, u = \nabla u.$$

- The directional derivative of the scalar field $u(x,y,z)$ in the point A along the unit vector $\mathbf{s}$ can be written in the form

$$\frac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}} = \frac{\partial u(A)}{\partial x}s_1 + \frac{\partial u(A)}{\partial y}s_2 + \frac{\partial u(A)}{\partial z}s_3 = \mathbf{grad}\, u \cdot \mathbf{s}.$$

Theorem (Properties of the gradient)

1. Gradient of the scalar field $u(x,y,z)$ is perpendicular to the contours of the scalar field $u(x,y,z)$ in each point $A \in \Omega$.
2. Gradient of the scalar field $u(x,y,z)$ points in the direction of the greatest increase of the scalar field $u(x,y,z)$. The opposite direction is the greatest rate of decrease of the scalar field $u(x,y,z)$.
3. The value of the greatest increase of the scalar field $u(x,y,z)$ is equal to $|\,\mathbf{grad}\, u|$.

Example 46

Find the directional derivative of the scalar field $u = 3x^2 - 4y^3 + 2z^4$ in the point $A = [1,2,1]$ along the vector $\mathbf{s} = \mathbf{AB}$, while $B = [4,6,6]$.

We need to find the values of the partial derivatives of the scalar field u in the point A.

$$\frac{\partial u}{\partial x} = 6x, \qquad \frac{\partial u}{\partial y} = -12y^2, \qquad \frac{\partial u}{\partial z} = 8z^3,$$

$$\frac{\partial u(A)}{\partial x} = 6, \qquad \frac{\partial u(A)}{\partial y} = -48, \qquad \frac{\partial u(A)}{\partial z} = 8.$$

To find a unit vector **s** in the direction of the **AB** vector we need to calculate

$$\mathbf{AB} = B - A = (3, 4, 5),$$

$$|\mathbf{AB}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = 5\sqrt{2},$$

$$\mathbf{s} = \frac{\mathbf{AB}}{|\mathbf{AB}|} = \left(\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}\right) = \left(\frac{3\sqrt{2}}{10}, \frac{2\sqrt{2}}{5}, \frac{\sqrt{2}}{2}\right).$$

By using formula for the directional derivative we obtain

$$\frac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}} = 6 \cdot \frac{3\sqrt{2}}{10} - 48 \cdot \frac{2\sqrt{2}}{2} + 8 \cdot \frac{\sqrt{2}}{2} = -\frac{67}{5}\sqrt{2}.$$

Example 47

Find the gradient of the scalar field $u = x^2 + y^2 + z^2 - 2xy + 2xz + 2yz$, the unit direction **s** of the greatest rate of increase of the field in the point $A = [1, 2, 1]$ and the greatest value of directional derivative of the scalar field u in the point A.

We calculate the partial derivatives of the scalar field $u(x, y, z)$.

$$\frac{\partial u}{\partial x} = 2x - 2y + 2z, \qquad \frac{\partial u}{\partial y} = 2y - 2x + 2z, \qquad \frac{\partial u}{\partial z} = 2z + 2x + 2y.$$

Therefore the gradient vector is in the form

$$\mathbf{grad}\, u = (2x - 2y + 2z, 2y - 2x + 2z, 2z + 2x + 2y).$$

Because the gradient always points to the direction of the greatest increase of the scalar field u, we can calculate the unit vector **s** by

$$\mathbf{grad}\, u(A) = (0, 4, 8),$$

$$|\,\mathbf{grad}\, u(A)| = \sqrt{80} = 4\sqrt{5},$$

$$\mathbf{s} = \frac{\mathbf{grad}\, u(A)}{|\,\mathbf{grad}\, u(A)|} = \left(0, \frac{4}{4\sqrt{5}}, \frac{8}{4\sqrt{5}}\right) = \left(0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right).$$

Now we are able to obtain the directional derivative $\dfrac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}}$ by

$$\frac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}} = \mathbf{grad}\, u(A) \cdot \mathbf{s} = (0, 4, 8) \cdot \left(0, \frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$$

$$= 0 + \frac{4\sqrt{5}}{5} + \frac{16\sqrt{5}}{5} = 4\sqrt{5}.$$

We can compare the results and confirm that for the direction of the greatest increase of the scalar field u it holds

$$\frac{\mathrm{d}u(A)}{\mathrm{d}\mathbf{s}} = |\,\mathbf{grad}\, u(A)|.$$

Exercise 48

Calculate directional derivative of scalar field u in the point A along the unit vector $\mathbf{s}$:

a) $u = 5x^4 - 4xy + 2y - 7, \quad A = [1,1], \quad \mathbf{s} = -\mathbf{i},$

b) $u = \sqrt{x^2 + y^2}, \quad A = [3,4], \quad \mathbf{s} \parallel \mathbf{v} = (4,-3).$

Exercise 49

a) Find the points where gradient of the scalar field $u = x^2 + 2xy + 4y^2 + z^2 - 4z$ is equal to zero.

b) Find the direction of the greatest rate of increase of the scalar field $u = x^2 + y^2 + z^2 - 1$ in the point $A = [0,-2,1]$.

3.3 Vector field

We often use vector fields to describe different physical phenomena. Vector field assigns to each point $X = [x,y,z]$ in the domain Ω the only vector $\mathbf{f}$, whose components are real functions $P(x,y,z), Q(x,y,z), R(x,y,z)$.

Definition

Vector field on the domain $\Omega \subset \mathbb{R}^3$ is given by a vector function

$$\mathbf{f}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}.$$

Definition

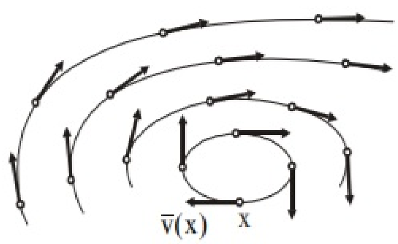
Let functions $P(x,y,z), Q(x,y,z), R(x,y,z)$ be continuous on the domain Ω and $X = [x,y,z] \in \Omega$ be an arbitrary point. The vector field $\mathbf{f} = (P(X), Q(X), R(X))$ is said to be **conservative** if and only if there exist a scalar field Φ on Ω such that

$$\mathbf{f} = \left(\frac{\partial \Phi(X)}{\partial x}, \frac{\partial \Phi(X)}{\partial y}, \frac{\partial \Phi(X)}{\partial z}\right) = \mathbf{grad}\,\Phi(x,y,z).$$

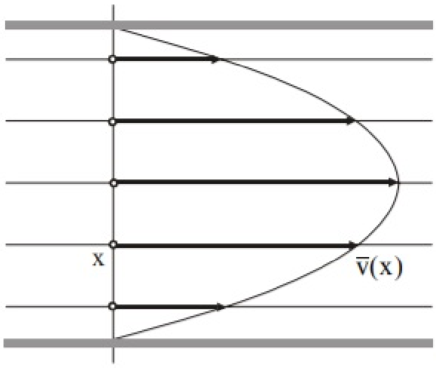
The scalar field Φ is called a **scalar potential** of a vector function $\mathbf{f}$.

To describe a vector field we use lines of force, flow lines, etc. The vector field $\mathbf{f}(X)$ always points in the direction of the tangent of such lines in each point $X \in \Omega$. See figures where you can find

- peripheral velocity of the rotational movement of the solid body



- velocity of laminar flow of the fluid



Definition

Let vector field be given by vector function

$$\mathbf{f}(X) = P(X)\,\mathbf{i} + Q(X)\,\mathbf{j} + R(X)\,\mathbf{k},$$

while functions $P(X), Q(X), R(X)$ are continuous and have partial derivatives on Ω.

- **Divergence of the vector field** $\mathbf{f}(X)$ is defined as a scalar field

$$\operatorname{div} \mathbf{f}(X) = \nabla \cdot \mathbf{f}(X) = \frac{\partial P(X)}{\partial x} + \frac{\partial Q(X)}{\partial y} + \frac{\partial R(X)}{\partial z}.$$

- Vector field , where for all $X \in \Omega$ holds $\operatorname{div} \mathbf{f}(X) = 0$ is called **solenoidal** (divergence-free).

- Points $X \in \Omega$, where $\operatorname{div} \mathbf{f}(X) > 0$ are called **sources**.

- Points $X \in \Omega$, where $\operatorname{div} \mathbf{f}(X) < 0$ are called **sinks**.

- **Curl of the vector field** $\mathbf{f}(X)$ is defined as a vector field

$$\mathbf{curl}\,\mathbf{f}(X) = \nabla \times \mathbf{f}(X) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(X) & Q(X) & R(X) \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

- Vector field, where for all $X \in \Omega$ holds $\mathbf{curl}\,\mathbf{f}(X) = 0$ is called **irrotational** (curl-free).

We can clarify the meaning of the divergence and the curl of vector field on the vector field of velocity $\mathbf{v}(x, y, z)$ of stationary flow of the fluid. Divergence of the vector field $\mathbf{v}$ in point A describes the volume of the fluid that flows out from unit of volume in unit of time in the neighbourhood of point A, i.e. intensity of the source of unit volume. Curl of the vector field $\mathbf{v}$ in point A defines the direction of the axis around which the fluid rotates in the neighbourhood of point A.

Theorem

Vector field $\mathbf{f}(X) = P(X)\,\mathbf{i} + Q(X)\,\mathbf{j} + R(X)\,\mathbf{k}$ is conservative on Ω if and only if it is irrotational on Ω, i.e. $\mathbf{curl}\,\mathbf{f}(X) = \mathbf{o}$.

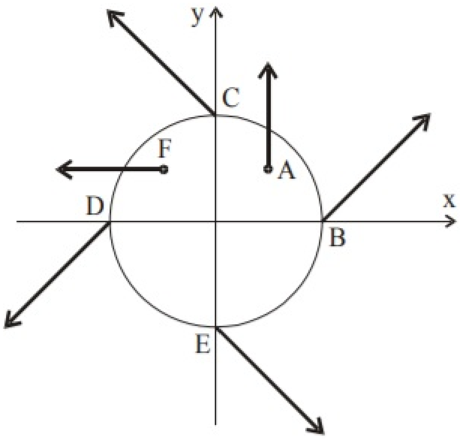
Example 50

Represent the vector field $\mathbf{f}(x,y) = (x-y)\,\mathbf{i} + (x+y)\,\mathbf{j}$ given on $\Omega\colon x^2+y^2 \leq 4$.

To represent the vector field we can choose some points in the domain Ω and calculate appropriate values of the vector field $\mathbf{f}$.

$$\begin{aligned}
A &= [1,1]: && \mathbf{f}(A) = 0\cdot\mathbf{i} + 2\cdot\mathbf{j} = (0,2)\\
B &= [2,0]: && \mathbf{f}(B) = 2\cdot\mathbf{i} + 2\cdot\mathbf{j} = (2,2)\\
C &= [0,2]: && \mathbf{f}(C) = -2\cdot\mathbf{i} + 2\cdot\mathbf{j} = (-2,2)\\
D &= [-2,0]: && \mathbf{f}(D) = -2\cdot\mathbf{i} - 2\cdot\mathbf{j} = (-2,-2)\\
E &= [0,-2]: && \mathbf{f}(E) = 2\cdot\mathbf{i} - 2\cdot\mathbf{j} = (2,-2)\\
F &= [-1,1]: && \mathbf{f}(F) = -2\cdot\mathbf{i} + 0\cdot\mathbf{j} = (-2,0)
\end{aligned}$$

The representation of the vector field is visible on following figure.



Example 51

Find out if the vector field $\mathbf{f}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$ is

- solenoidal,
- irrotational.
- Find its scalar potential Φ if exists.

For the vector field $\mathbf{f}$ it holds

$$P = x^2, \quad Q = y^2, \quad R = z^2,$$

$$\frac{\partial P}{\partial x} = 2x, \quad \frac{\partial P}{\partial y} = 2y, \quad \frac{\partial P}{\partial z} = 2z.$$

- According to the definition of divergence we obtain

$$\operatorname{div}\mathbf{f}(x,y,z) = 2x + 2y + 2z,$$

therefore the vector field is not solenoidal. Points, where $2x+2y+2z > 0$ are sources, while there are sinks in points where $2x+2y+2z < 0$. For example the point $A = [1,1,1]$ is source because $\operatorname{div} \mathbf{f}(A) = 6 > 0$. The point $B = [-1,-1,-1]$ is sink because $\operatorname{div} \mathbf{f}(B) = -6 < 0$.

- Based on the definition of curl we calculate

$$\mathbf{curl}\,\mathbf{f}(X) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix} = \left(\frac{\partial\,(z^2)}{\partial y} - \frac{\partial\,(y^2)}{\partial z}, \frac{\partial\,(x^2)}{\partial z} - \frac{\partial\,(z^2)}{\partial x}, \frac{\partial\,(y^2)}{\partial x} - \frac{\partial\,(x^2)}{\partial y} \right) = \mathbf{o}.$$

Therefore, the given field is irrotational and it is also conservative.

- We use properties $P = \frac{\partial \Phi}{\partial x}$, $Q = \frac{\partial \Phi}{\partial y}$, $R = \frac{\partial \Phi}{\partial z}$ from the definition of the scalar potential. Hence

$$\Phi = \int P \,\mathrm{d}x = \int x^2 \,\mathrm{d}x = \frac{x^3}{3} + K_1(y,z),$$

where $K_1(y,z)$ is an arbitrary function depending on variables y, z. To find it we derivate the scalar potential Φ with respect to y and realise that

$$\frac{\partial K_1(y,z)}{\partial y} = y^2$$

from which we obtain

$$K_1(y,z) = \int y^2 \,\mathrm{d}y = \frac{y^3}{3} + K_2(z),$$

where $K_2(z)$ is an arbitrary function depending only on variable z, which must fulfil $K_2'(z) = z^2$ based on the partial derivative of potential Φ with respect to z. Therefore

$$K_2(z) = \int z^2 \,\mathrm{d}z = \frac{z^3}{3} + C,$$

where C is an arbitrary real constant. Finally, we obtained scalar potential in the form

$$\Phi(X) = \frac{x^2 + y^2 + z^2}{3} + C.$$

Example 52

Vector field of the force $\mathbf{F}$ in each point $X = [x,y,z]$ points to the coordinates origin and its magnitude is equal to $|\mathbf{F}| = \frac{1}{\rho^2}$, where ρ is the distance between the point and the coordinate origin. Find out if the field is conservative.

The force $\mathbf{F}$ has the same direction as the position vector $\mathbf{r}$ of an arbitrary point X,

$$\mathbf{r} = \mathbf{OX} = X - O = (x,y,z)$$

but the opposite orientation. Therefore

$$\mathbf{F} = (-cx, -cy, -cz),$$

where $c > 0$ is an arbitrary constant. The distance of the point $X = [x, y, z]$ is equal to

$$\rho = |OX| = \sqrt{x^2 + y^2 + z^2}.$$

The magnitude of the force is given and is equal to

$$|\mathbf{F}| = \frac{1}{\rho^2} = \frac{1}{x^2 + y^2 + z^2}$$

from which we obtain

$$c = \frac{1}{\sqrt{(x^2 + y^2 + z^2)^3}}.$$

We have found the components of the force vector

$$\mathbf{F} = -\frac{x}{\sqrt{(x^2 + y^2 + z^2)^3}}\mathbf{i} - \frac{y}{\sqrt{(x^2 + y^2 + z^2)^3}}\mathbf{j} - \frac{z}{\sqrt{(x^2 + y^2 + z^2)^3}}\mathbf{k}.$$

According to the definition of the curl we calculate

$$\begin{aligned} \mathbf{curl}\,\mathbf{F} = & \left(\frac{3yz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} - \frac{3yz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} \right) \mathbf{i} \\ & + \left(\frac{3xz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} - \frac{3xz\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} \right) \mathbf{j} \\ & + \left(\frac{3xy\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} - \frac{3xy\sqrt{x^2 + y^2 + z^2}}{(x^2 + y^2 + z^2)^3} \right) \mathbf{k} = \mathbf{o}. \end{aligned}$$

The vector field is irrotational and therefore it is also conservative.

Exercise 53

Find the divergence and the curl of the vector field $\mathbf{f}$.

a) $\mathbf{f}(x, y, z) = x^2yz\,\mathbf{i} + xy^2z\,\mathbf{j} + xyz^2\,\mathbf{k}$

b) $\mathbf{f}(x, y, z) = \mathbf{grad}\left(x^3 + y^3 + z^3\right)$

Exercise 54

Find out if following vector field $\mathbf{f}$ is solenoidal, irrotational and conservative, find its scalar potential Φ if exists:

a) $\mathbf{f}(x, y, z) = (y + z)\,\mathbf{i} + (x + z)\,\mathbf{j} + (x + y)\,\mathbf{k}$,

b) $\mathbf{f}(x, y, z) = \mathbf{grad}(xyz)$.

4 Line integral

4.1 The curve and its parametrization

Definition

Let $x = x(t)$, $y = y(t)$, $z = z(t)$ be continuous functions for $t \in [a, b]$. The **curve** k with parametrical equations

$$\begin{aligned} x &= x(t), \\ y &= y(t), \\ z &= z(t), \quad t \in [a, b] \end{aligned}$$

is called **positively oriented with respect to the parameter** t, if and only if its points are ordered so that for arbitrary values $t_1, t_2 \in [a, b]$, $t_1 < t_2$, the point $M_1 = [x(t_1), y(t_1), z(t_1)]$ lies before the point $M_2 = [x(t_2), y(t_2), z(t_2)]$, i.e.

$$\forall t_1, t_2 \in [a, b] : t_1 < t_2 \Leftrightarrow M_1 \prec M_2.$$

Reversely,

$$\forall t_1, t_2 \in [a, b] : t_1 < t_2 \Leftrightarrow M_2 \prec M_1,$$

the curve is called **negatively oriented with respect to the parameter** t.

Remark

The symbol $\prec$ means "precedes" or "lies before".

Definition

If the curve k is positively oriented with respect to the parameter $t \in [a, b]$, then the point $A = [x(a), y(a), z(a)]$ is called the **starting point** of the curve and the point $B = [x(b), y(b), z(b)]$ is called the **ending point** of the curve.

Definition

Let curve k is given by parametrical equations

$$\begin{aligned} x &= x(t), \\ y &= y(t), \\ z &= z(t), \quad t \in [a, b] \end{aligned}$$

with starting point $A = [x(a), y(a), z(a)]$ and ending point $B = [x(b), y(b), z(b)]$.

- The curve is called **closed**, if $A \equiv B$.
- The curve is called **smooth** on $[a, b]$, if there exists continuous derivatives of para-

metrical equations

$$\begin{aligned} \dot{x} &= \dot{x}(t), \\ \dot{y} &= \dot{y}(t), \\ \dot{z} &= \dot{z}(t) \end{aligned}$$

and $\forall t \in [a,b] : \ (\dot{x}(t), \dot{y}(t), \dot{z}(t)) \neq (0,0,0)$.

- The curve is called **piecewise smooth** on $[a,b]$, if it is smooth on $[a,b]$ except for a finite number of points $t_i \in [a,b]$, $i = 1, \dots, n$.
- The curve is called **simple** on $[a,b]$, if it doesn't intersect itself, i.e.

$$\forall t_1, t_2 \in (a,b) : \quad t_1 \neq t_2 \Rightarrow M_1 \neq M_2.$$

Example 55

Write parametrization of the line segment $\overline{AB}$, where $A = [0,0]$, $B = [1,1]$.

There are infinitely many possibilities how to write down a parametrization. For example:

1. If we consider the given segment as a part of graph of function $y = x$, we can put $t = x = y$ and obtain

$$\begin{aligned} x &= t, \\ y &= t, \quad t \in [0,1]. \end{aligned}$$

   The curve is positively oriented. For $t = 0$ we obtain $A = [x(0), y(0)] = [0,0]$ and analogically for $t = 1$ we obtain $B = [x(1), y(1)] = [1,1]$.

2. It is not necessary to keep $x = t$. We can use parametrization

$$\begin{aligned} x &= r - 1, \\ y &= r - 1, \quad r \in [1,2]. \end{aligned}$$

   which is also positively oriented. We obtain $A = [x(1), y(1)] = [0,0]$, while $B = [x(2), y(2)] = [1,1]$.

3. If we use following parametrization

$$\begin{aligned} x &= -s, \\ y &= -s, \quad s \in [-1,0]. \end{aligned}$$

   The curve is then negatively oriented. In such situation $B = [x(-1), y(-1)] = [1,1]$, while $A = [x(0), y(0)] = [0,0]$.

4.1.1 Parametrization of the line segment

Parametrizations of the line segment between points $A = [a_1, a_2, a_3]$, $B = [b_1, b_2, b_3]$ are in the form

$$\begin{aligned} x &= a_1 + u_1 \cdot t, \\ y &= a_2 + u_2 \cdot t, \\ z &= a_3 + u_3 \cdot t, \quad t \in [0,1]. \end{aligned}$$

where $\mathbf{u} = \mathbf{AB} = (u_1, u_2, u_3)$ is the vector parallel to the line segment AB.

Example 56

Write parametrization of the line segment between points $A = [1, 2, \pi]$ and $B = [8, -3, 0]$.

We compute vector

$$\mathbf{AB} = B - A = (7, -5, -\pi).$$

The parametrical equations are

$$\begin{aligned} x &= 1 + 7t, \\ y &= 2 - 5t, \\ z &= \pi - \pi t, \quad t \in [0,1]. \end{aligned}$$

4.1.2 Parametrization of the circle

Parametrical equations of the circle

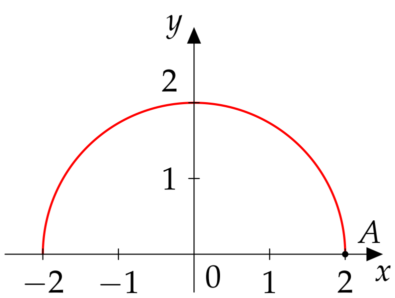
$$(x-m)^2 + (y-n)^2 = r^2, \quad r > 0,$$

with the center in the point $C = [m, n]$ and radius r are in the form

$$\begin{aligned} x &= m + r\cos t, \\ y &= n + r\sin t, \quad t \in [0, 2\pi]. \end{aligned}$$

Example 57

Compute parametrization of the circle with the center in the origin and radius $r = 2$ for $y \geq 0$. The starting point of the curve is $A = [2, 0]$.



The parametrical equations are

$$\begin{aligned} x &= 2\cos t, \\ y &= 2\sin t, \quad t \in [0, \pi]. \end{aligned}$$

The curve is positively oriented with respect to the parameter t.

We can also express variable y from equation

$$x^2 + y^2 = 4$$

and obtain

$$y = \pm\sqrt{4 - x^2}.$$

For $y > 0$ we consider only

$$y = \sqrt{4 - x^2}.$$

By putting $s = x$ we then obtain parametrical equations of the given curve in the form

$$\begin{aligned} x &= s, \\ y &= \sqrt{4 - s^2}, \quad s \in [-2, 2]. \end{aligned}$$

The curve is negatively oriented with respect to parameter s.

4.1.3 Parametrization of the ellipse

Parametrical equations of the ellipse

$$\frac{(x-m)^2}{a^2} + \frac{(y-n)^2}{b^2} = 1, \quad a, b > 0,$$

with the center in the point $[m, n]$ and semi-axis a, b are in the form

$$\begin{aligned} x &= m + a\cos t, \\ y &= n + b\sin t, \quad t \in [0, 2\pi]. \end{aligned}$$

Example 58

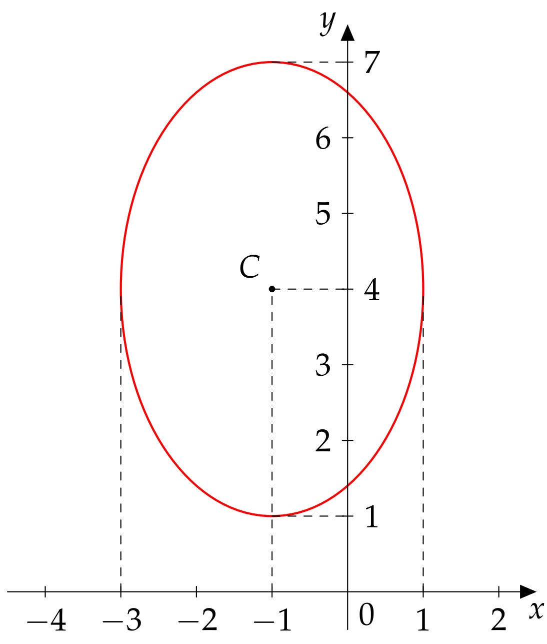
Compute parametrization of the curve $9x^2 + 4y^2 + 18x - 32y + 37 = 0$.

We can find the center of the ellipse and sizes of the semi-axes by following calculation

$$\begin{aligned} 9x^2 + 4y^2 + 18x - 32y + 37 &= 0 \\ 9\left(x^2 + 2x\right) + 4\left(y^2 - 8y\right) &= -37 \\ 9\left(x^2 + 2x + 1\right) - 9 + 4\left(y^2 - 8y + 16\right) - 64 &= -37 \\ 9(x+1)^2 + 4(y-4)^2 &= 36 \\ \frac{(x+1)^2}{4} + \frac{(y-4)^2}{9} &= 1 \end{aligned}$$

The center of the ellipse is in point $C = [-1, 4]$ and semi-axis are $a = 2$, $b = 3$ and the parametrical equations are

$$\begin{aligned} x &= -1 + 2\cos t, \\ y &= 4 + 3\sin t, \quad t \in [0, 2\pi]. \end{aligned}$$



4.2 Line integral of a scalar field

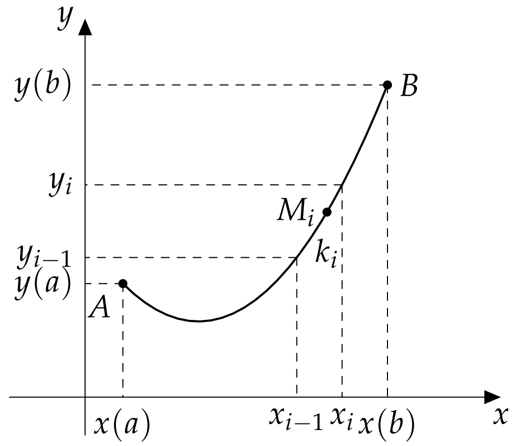
We need to divide our domain (the curve k) into small elements. Let us consider the simple smooth curve k with parametrization

$$\begin{aligned} x &= x(t), \\ y &= y(t), \\ z &= z(t), \quad t \in [a,b] \end{aligned}$$

positively oriented with respect to the parameter t. We divide interval $[a,b]$ by sequence of points

$$a = t_0 < t_1 < \cdots < t_n = b$$

into n partial curves $k_1, k_2, \ldots, k_n$ according to the figure



For each $i = 1, \ldots, n$ we denote by Δs_i the length of each element k_i and we choose an arbitrary point $M_i = [x(t_i), y(t_i), z(t_i)]$ in each element k_i. The curve k lies within a domain Ω and we consider a bounded continuous scalar function $u(X) = u(x,y,z)$ defined for each $X \in \Omega$. Now we can create the sum of products

$$\sum_{i=1}^{n} u(M_i) \cdot \Delta s_i = \sum_{i=1}^{n} u(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i$$

and define the line integral of a scalar field.

Definition

If there exists

$$\lim \sum_{i=1}^{n} u(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i$$

for $n \to \infty$ and $\Delta s_i \to 0$ we call it the **line integral of a scalar field** $u(x, y, z)$ **along the curve** k and denote it

$$\int_k u(x, y, z)\, \mathrm{d}s.$$

Theorem (Properties of the line integral of a scalar field)

1. $\displaystyle\int_k c\, u(X)\, \mathrm{d}s = c \int_k u(X)\, \mathrm{d}s,$
2. $\displaystyle\int_k (u(X) + v(X))\, \mathrm{d}s = \int_k u(X)\, \mathrm{d}s + \int_k v(X)\, \mathrm{d}s,$
3. $\displaystyle\int_k u(X)\, \mathrm{d}s = \int_{k_1} u(X)\, \mathrm{d}s + \int_{k_2} u(X)\, \mathrm{d}s,$

where $c \in \mathbb{R}$, k_1, k_2 are non-overlapping curves such that curve k fulfils $k = k_1 \cup k_2$ and $u(X), v(X)$ are bounded continuous scalar functions for all $X \in \Omega$ that contains the curve k.

Remark

The line integral of the scalar field doesn't depend on the orientation of the curve, because the lengths of all components Δs_i are always positive.

The element of the curve $\mathrm{d}s$ in the three-dimensional space is the body diagonal of the rectangular hexahedron with sides $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$. Therefore we obtain

$$\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2} = \sqrt{(\dot{x}(t)\,\mathrm{d}t)^2 + (\dot{y}(t)\,\mathrm{d}t)^2 + (\dot{z}(t)\,\mathrm{d}t)^2}$$

$$= \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2}\,\mathrm{d}t.$$

The line integral of the scalar field can then be written in the form

$$\int_k u(x,y,z)\,\mathrm{d}s = \int_a^b u(x(t),y(t),z(t))\sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2+(\dot{z}(t))^2}\,\mathrm{d}t.$$

Example 59

Calculate the line integral $\int_k (x+z)\,\mathrm{d}s$ along the line segment between points $A=[1,2,3]$, $B=[3,2,1]$.

We create the parametrization of the line segment

$$\begin{aligned} x &= 1+2t, \\ y &= 2, \\ z &= 3-2t, \quad t\in[0,1] \end{aligned}$$

and calculate its derivatives

$$\dot{x}=2, \qquad \dot{y}=0, \qquad \dot{z}=-2.$$

We express the element $\mathrm{d}s$

$$\mathrm{d}s = \sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2+(\dot{z}(t))^2}\,\mathrm{d}t = \sqrt{8}\,\mathrm{d}t = 2\sqrt{2}\,\mathrm{d}t$$

to calculate the line integral

$$\int_k (x+z)\,\mathrm{d}s = \int_0^1 (1+2t+3-2t)\cdot 2\sqrt{2}\,\mathrm{d}t = 8\sqrt{2}\int_0^1 \mathrm{d}t = 8\sqrt{2}.$$

If we consider just the two-dimensional problem, i.e. the curve is in xy-plane, the element

$$\mathrm{d}s = \sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2}\,\mathrm{d}t$$

and the line integral of the scalar field is then in the form

$$\int_k u(x,y)\,\mathrm{d}s = \int_a^b u(x(t),y(t))\sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2}\,\mathrm{d}t.$$

Example 60

Calculate the line integral $\int_k y^2\,\mathrm{d}s$, where k is a circle with the center in the origin of coordinates and radius 2.

The parametrical equations of the circle are

$$\begin{aligned} x &= 2\cos t, \\ y &= 2\sin t, \quad t \in [0, 2\pi]. \end{aligned}$$

We calculate derivatives

$$\begin{aligned} \dot{x} &= -2\sin t, \\ \dot{y} &= 2\cos t \end{aligned}$$

and the element of the curve

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\,dt = \sqrt{4\sin^2 t + 4\cos^2 t}\,dt = \sqrt{4}\,dt = 2\,dt.$$

We are able to calculate the integral

$$\int\limits_k y^2\,ds = \int\limits_0^{2\pi} 4\sin^2 t \cdot 2\,dt = 8\int\limits_0^{2\pi} \frac{1}{2}\left(1 - \cos(2t)\right)\,dt$$

$$= 4\left[t - \frac{1}{2}\sin(2t)\right]_0^{2\pi} = 8\pi.$$

Example 61

Calculate the line integral $\int\limits_k y\,ds$, where k is a part of the function $y = x^3$ between points $A = [0, 0]$, $B = [1, 1]$.

Parametrical equations of function $y = x^3$, $x \in [0, 1]$ are

$$\begin{aligned} x &= t, \\ y &= t^3, \quad t \in [0, 1]. \end{aligned}$$

We need to express a derivative of parametrical equations

$$\begin{aligned} x &= 1, \\ y &= 3t^2 \end{aligned}$$

an element of the curve

$$ds = \sqrt{1 + (3t^2)^2}\,dt = \sqrt{1 + 9t^4}\,dt$$

to calculate the integral

$$\int\limits_k y\,ds = \int\limits_0^1 t^3\sqrt{1 + 9t^4}\,dt.$$

In such integral we can use substitution

$$\begin{aligned} 1 + 9t^4 &= z \\ 36t^3\,dt &= dz \end{aligned}$$

to obtain

$$\int\limits_k y\,ds = \frac{1}{36}\int\limits_1^{10} \sqrt{z}\,dz = \frac{1}{36} \cdot \frac{2}{3}\left[\sqrt{z^3}\right]_1^{10} = \frac{1}{54}\left(10\sqrt{10} - 1\right).$$

Exercise 62

Compute the line integrals of scalar fields along given curves.

a) $\displaystyle\int_k \frac{z^2}{x^2+y^2}\,\mathrm{d}s$, k is one thread of the spiral $x=\cos t$, $y=\sin t$, $z=t$, $t\in[0,2\pi]$

b) $\displaystyle\int_k x\,\mathrm{d}s$, k is a line segment between points $A=[0,0]$, $B=[1,2]$

c) $\displaystyle\int_k x^2\,\mathrm{d}s$, k is an upper half of the circle $x^2+y^2=a^2$, $a>0$

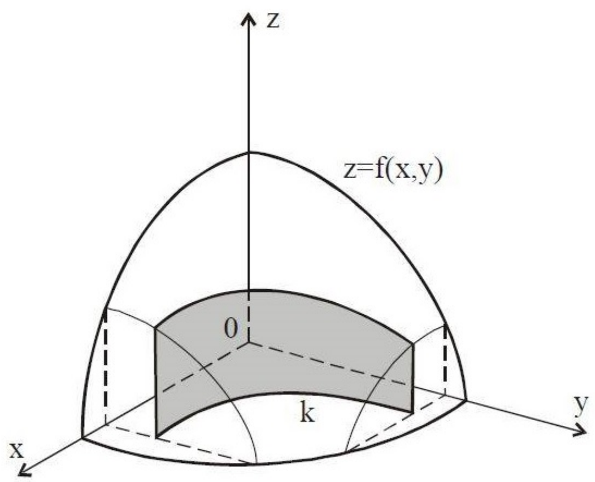
d) $\displaystyle\int_k x^2\,\mathrm{d}s$, $k: y=\ln x$, $x\in[1,3]$

4.2.1 Practical applications of line integral of the scalar field

Area of a cylindrical region

Let function $f(x,y)\geq 0$ be continuous on a domain Ω that contains the curve k. We consider **the cylindrical surface between the plane** $z=0$ **and** $z=f(x,y)$ **above the curve** k, see figure. The area of such a surface is

$$A=\int_k f(x,y)\,\mathrm{d}s.$$



Example 63

Calculate the area of the cylindrical surface $x^2+y^2=r^2$ bounded by $z\geq 0$ and $z\leq x$.

The curve k is a part of the circle $x^2+y^2=r^2$ for $x\geq 0$. Therefore, the parametrization of the curve k is in the form

$$\begin{aligned} x &= r\cos t, \\ y &= r\sin t, \quad t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]. \end{aligned}$$

We express derivatives of the parametric equations

$$\begin{aligned} \dot{x} &= -r \sin t, \\ \dot{y} &= r \cos t \end{aligned}$$

and the element of the curve is then given by

$$\mathrm{d}s = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\,\mathrm{d}t = \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\,\mathrm{d}t = r\,\mathrm{d}t.$$

Now we can calculate the area of given cylindrical surface

$$A = \int_k x\,\mathrm{d}s = \int_{-\pi/2}^{\pi/2} r \cos t \cdot r\,\mathrm{d}t = r^2 [\sin t]_{-\pi/2}^{\pi/2} = 2r^2.$$

Exercise 64

Calculate the area of cylindrical surfaces bounded by given conditions:

a) $x^2 + y^2 = r^2$, $z \geq 0$, $z \leq \dfrac{xy}{2r}$, $x \geq 0$, $y \geq 0$,

b) $9y^2 = 4(x-1)^3$, $z \geq 0$, $z \leq 2 - \sqrt{x}$,

c) $y^2 = 2x$, $z \geq 0$, $z \leq \sqrt{2x - 4x^2}$,

d) $y = \dfrac{3}{8}x^2$, $z \geq 0$, $z \leq x$, $x \geq 0$, $y \leq 6$.

Length of a curve

Let k be simple, piecewise smooth curve. **The length of the curve** is numerically equal to the area of the cylindrical surface above the curve k that is bounded by planes $z = 0$ and $z = 1$. Hence, by letting $f(x, y) = 1$ in formula for area of a cylindrical region $A = \int_k f(x,y)\,\mathrm{d}s$ we obtain

$$L = \int_k \mathrm{d}s.$$

Remark

The length of the curve k in three dimensional space can be calculated using the same formula

$$L = \int_k \mathrm{d}s.$$

Example 65

Calculate the length of one period of the cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $t \in [0, 2\pi]$, $a > 0$.

We need to calculate the derivatives of the parametric equations of the cycloid

$$\begin{aligned} \dot{x} &= a(1-\cos t), \\ \dot{y} &= a\sin t. \end{aligned}$$

Further, we use them to express the element of the curve

$$ds = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\,dt = \sqrt{a^2(1-\cos t)^2 + a^2\sin^2 t}\,dt$$

$$= a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt = a\sqrt{2 - 2\cos t}\,dt = \sqrt{2}a\sqrt{1-\cos t}\,dt.$$

Now, we need to use trigonometric identity $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and obtain

$$ds = \sqrt{2}a\sqrt{2\sin^2\frac{t}{2}}\,dt = 2a\sin\frac{t}{2}\,dt.$$

Finally, we are able to calculate the length of the curve

$$L = \int_k ds = \int_0^{2\pi} 2a\sin\frac{t}{2}\,dt = 2a\left[-2\cos\frac{t}{2}\right]_0^{2\pi} = -4a\cdot(-1-1) = 8a.$$

Exercise 66

Calculate the lengths of the given curves.

a) cardioid with parametrical equations $x = 2a\cos t - a\cos 2t, y = 2a\sin t - a\sin 2t$, $t \in [0, 2\pi]$, $a > 0$

b) $y = \frac{1}{2}\ln x,\ z = \frac{1}{2}x^2,\ x \in [1, 2]$

c) $y = 1 - \ln\cos x,\ x \in \left[0, \frac{\pi}{4}\right]$

d) $y = \frac{1}{2}x^2,\ z = \frac{1}{6}x^3,\ x \in [0, 1]$

Mass of a curve

Let k be a simple, piecewise smooth curve and continuous function $\rho(x, y, z) > 0$ be its linear density. **The mass of a curve** (e.g. mass of a wire) is given by the line integral of a scalar field

$$m = \int_k \rho(x, y, z)\,ds.$$

Example 67

Calculate the mass of one thread of the screw line $k: x = \cos t, y = \sin t, z = t, t \in [0, 2\pi]$ if its density is given by $\rho = x^2 + y^2 + z^2$.

Derivatives of the parametric equations are in the form

$$\begin{aligned} \dot{x} &= -\sin t, \\ \dot{y} &= \cos t, \\ \dot{z} &= 1. \end{aligned}$$

We express element of the curve

$$\mathrm{d}s = \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2 + (\dot{z}(t))^2}\,\mathrm{d}t = \sqrt{\sin^2 t + \cos^2 t + 1^2}\,\mathrm{d}t = \sqrt{2}\,\mathrm{d}t.$$

Now we can use the line integral and calculate the mass of the given curve

$$m = \int_k \left(x^2 + y^2 + z^2\right)\,\mathrm{d}s = \sqrt{2}\int_0^{2\pi} \left(\cos^2 t + \sin^2 t + t^2\right)\,\mathrm{d}t = \sqrt{2}\int_0^{2\pi} \left(t^2 + 1\right)\,\mathrm{d}t$$

$$= \sqrt{2}\left[\frac{t^3}{3} + t\right]_0^{2\pi} = \sqrt{2}\left(\frac{8\pi^3}{3} + 2\pi\right) = 2\sqrt{2}\pi\left(\frac{4}{3}\pi^3 + 1\right).$$

Exercise 68

a) Calculate the mass of one quarter of the circle $x = a\sin t,\ y = a\cos t,\ t \in \left[0, \frac{\pi}{2}\right]$ if the density in each point is equal to its y-coordinate.

b) Calculate the mass of the parabola $y = \frac{1}{2}x^2$ between the points $A = \left[1, \frac{1}{2}\right]$ and $B = [2, 2]$. The density $\rho = \frac{y}{x}$.

c) Calculate the mass of the curve $y = \ln x$, where $x \in [1, 2]$. The density at each point is equal to the square of its x-coordinate.

d) Calculate the mass of the catenary $y = \frac{a}{2}\left(\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{-\frac{x}{a}}\right)$ for $x \in [0, a]$, $a > 0$. The density $\rho = \frac{a}{y}$.

4.3 Line integral of a vector field

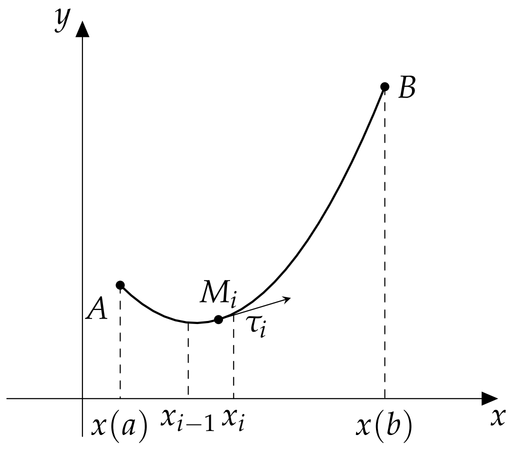
Let us consider the simple smooth curve k with parametrization

$$\begin{aligned} x &= x(t), \\ y &= y(t), \\ z &= z(t), \quad t \in [a, b] \end{aligned}$$

that is positively oriented with respect to the parameter t. The curve lies within the domain Ω.
We divide interval $[a, b]$ by sequence of points

$$a = t_0 < t_1 < \cdots < t_n = b$$

into n partial curves $k_1, k_2, \ldots, k_n$. We are also able to construct positively oriented unitary tangential vector $\boldsymbol{\tau}_i(M_i)$ at each point M_i according to the figure.



Furthermore, we consider bounded continuous vector field

$$\mathbf{F}(X) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k}$$

defined for each $X \in \Omega$. We create scalar products of the vector field and the positively oriented tangential vector of each element of the curve $\mathbf{F}(M_i) \cdot \Delta\mathbf{s}_i$, where $\Delta\mathbf{s}_i = \Delta s_i \boldsymbol{\tau}_i$. Now we can create sum of such products

$$\sum_{i=1}^{n} \mathbf{F}(M_i) \cdot \Delta\mathbf{s}_i = \sum_{i=1}^{n} \mathbf{F}(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i \boldsymbol{\tau}_i$$

and define the line integral of a vector field.

Definition

If there exists

$$\lim \sum_{i=1}^{n} \mathbf{F}(x(t_i), y(t_i), z(t_i)) \cdot \Delta s_i \boldsymbol{\tau}_i$$

for $n \to \infty$ and $\Delta s_i \to 0$, we call it the **line integral of a vector field $\mathbf{F}(x,y,z)$ along the curve** k_+ and denote it

$$\int_{k_+} \mathbf{F}(x,y,z) \cdot \mathrm{d}\mathbf{s},$$

where k_+ denotes curve k positively oriented with respect to parameter t. While k_- would denote curve k negatively oriented with respect to parameter t.

Remark

The line integral of the vector field depends on the orientation of the curve k, because coordinates of the unitary tangential vectors $\boldsymbol{\tau}_i$ depend on the orientation of the curve.

To derive the form of the line integral of the vector field, we need to express the inner product

$$\begin{aligned}\mathbf{F}(x,y,z)\cdot \mathrm{d}\mathbf{s} &= (P(x,y,z),Q(x,y,z),R(x,y,z))\cdot(\,\mathrm{d}x,\ \mathrm{d}y,\ \mathrm{d}z)\\ &= P(x,y,z)\,\mathrm{d}x + Q(x,y,z)\,\mathrm{d}y + R(x,y,z)\,\mathrm{d}z\end{aligned}$$

and the differentials $\mathrm{d}x = \dot{x}(t)\ \mathrm{d}t,\ \mathrm{d}y = \dot{y}(t)\ \mathrm{d}t,\ \mathrm{d}z = \dot{z}(t)\ \mathrm{d}t$ by using parametrization of the curve.

The line integral of the vector field then can be written in the form

$$\int_k \mathbf{F}(x,y,z)\cdot \mathrm{d}\mathbf{s} = \varepsilon \int_a^b \left[P(x(t),y(t),z(t))\dot{x}(t) + Q(x(t),y(t),z(t))\dot{y}(t)\right.$$
$$\left. + R(x(t),y(t),z(t))\dot{z}(t)\right]\ \mathrm{d}t,$$

where $\varepsilon = 1$ in case of positively oriented curve k with respect to the parameter t, while $\varepsilon = -1$ in case of negatively oriented curve k.

This way we transform the line integral of the vector field into the one-dimensional definite integral, similarly to the case of the line integral of a scalar field.

Theorem (Properties of the line integral of a vector field)

1. $\displaystyle\int_k c\,\mathbf{F}(X)\cdot \mathrm{d}\mathbf{s} = c\int_k \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s},$
2. $\displaystyle\int_k (\mathbf{F}(X)+\mathbf{G}(X))\cdot \mathrm{d}\mathbf{s} = \int_k \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s} + \int_k \mathbf{G}(X)\cdot \mathrm{d}\mathbf{s},$
3. $\displaystyle\int_k \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s} = \int_{k_1} \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s} + \int_{k_2} \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s},$
4. $\displaystyle\int_{k_+} \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s} = -\int_{k_-} \mathbf{F}(X)\cdot \mathrm{d}\mathbf{s}.$

where $c\in\mathbb{R}$, k_1, k_2 are non-overlapping curves such that curve k fulfils $k = k_1\cup k_2$ (considering the same orientation of these curves) and $\mathbf{F}(X), \mathbf{G}(X)$ are bounded continuous vector functions for all $X\in\Omega$ that contains the curve k.

Example 69

Calculate the line integral of the vector field $\mathbf{F} = (x,y,z)$ along the curve k, that is one thread of the spiral $x = 2\cos t,\ y = 2\sin t,\ z = 3t,\ t\in[0,2\pi]$. The curve is positively oriented with respect to the parameter t.

We express derivatives of parametrization equations

$$\begin{aligned}\dot{x} &= -2\sin t\\ \dot{y} &= 2\cos t\\ \dot{z} &= 3\end{aligned}$$

and we can calculate the integral

$$\int_k \mathbf{F}(x,y,z)\cdot\,\mathrm{d}\mathbf{s} = \int_k x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z$$

$$= \int_0^{2\pi} [2\cos t\cdot(-2\sin t) + 2\sin t\cdot 2\cos t + 3t\cdot 3]\,\mathrm{d}t$$

$$= \int_0^{2\pi} [-4\sin t\cos t + 4\sin t\cos t + 9t]\,\mathrm{d}t = \int_0^{2\pi} 9t\,\mathrm{d}t = \frac{9}{2}\left[t^2\right]_0^{2\pi} = 18\pi^2.$$

If we consider the two-dimensional problem, i.e. the curve is in xy-plane, the vector field $\mathbf{F} = (P(x,y), Q(x,y))$ and the tangential vector of the element

$$\mathrm{d}\mathbf{s} = (\,\mathrm{d}x,\,\mathrm{d}y) = (\dot{x}(t)\,\mathrm{d}t, \dot{y}(t)\,\mathrm{d}t),$$

the line integral of the vector field is then in the form

$$\int_k \mathbf{F}(x,y)\cdot d\mathbf{s} = \varepsilon\int_a^b [P(x(t),y(t))\dot{x}(t) + Q(x(t),y(t))\dot{y}(t)]\,\mathrm{d}t.$$

Example 70

Calculate the line integral $\int_k (x+y)\,\mathrm{d}x + (x-y)\,\mathrm{d}y$, where $k: y = \frac{1}{x}$, $x\in[2,3]$. The starting point of the curve k is $A = \left[2, \frac{1}{2}\right]$.

Parametrical equations of the curve k are

$$\begin{aligned} x &= t, \\ y &= \frac{1}{t}, \quad t\in[2,3]. \end{aligned}$$

The curve is positively oriented with respect to the parameter t. We calculate the derivatives

$$\begin{aligned} \dot{x} &= 1, \\ \dot{y} &= -\frac{1}{t^2} \end{aligned}$$

and the integral

$$\int_k (x+y)\,\mathrm{d}x + (x-y)\,\mathrm{d}y = \int_2^3 \left[\left(t+\frac{1}{t}\right) + \left(t-\frac{1}{t}\right)\cdot\left(-\frac{1}{t^2}\right)\right]\,\mathrm{d}t$$

$$= \int_2^3 \left[t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^3}\right] \mathrm{d}t = \int_2^3 \left(t + \frac{1}{t^3}\right) \mathrm{d}t = \left[\frac{t^2}{2} - \frac{1}{2t^2}\right]_2^3$$

$$= \frac{9}{2} - \frac{1}{18} - 2 + \frac{1}{8} = \frac{185}{72}.$$

Exercise 71

Compute the line integrals of vector fields along given curves.

a) $\int_k x\,\mathrm{d}x - y\,\mathrm{d}y + z\,\mathrm{d}z,\quad k$ is an oriented line segment $\overline{AB}$: $A = [1,1,1]$, $B = [4,3,2]$

b) $\int_k y\,\mathrm{d}x + x\,\mathrm{d}y,\quad k$ is one quarter of the circle $x = a\cos t$, $y = a\sin t$, $a > 0$, $t \in \left[0, \frac{\pi}{2}\right]$ with starting point $A\,[a,0]$

c) $\int_k (xy - 1)\ \mathrm{d}x + x^2 y\ \mathrm{d}y,\quad k$ is an arc of the ellipse $x = \cos t$, $y = 2\sin t$ from the starting point $A = [1,0]$ to the end point $B = [0,2]$

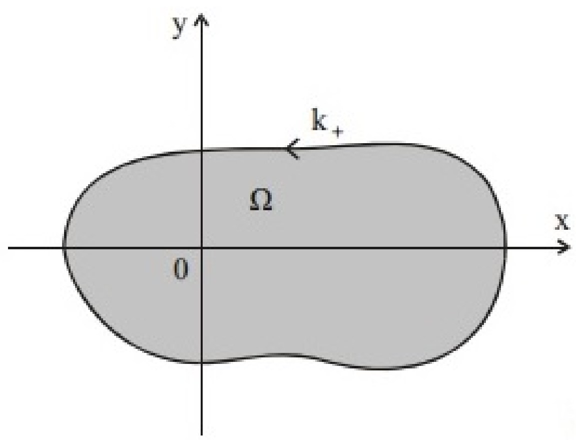
d) $\int_k xy\ \mathrm{d}x + (y - x)\ \mathrm{d}y,\quad k$ is a part of the parabola $y^2 = x$ from the starting point $A = [0,0]$ to the end point $B = [1,1]$

4.3.1 Green's theorem

Green's theorem expresses the relation between a line integral of a vector field along a plane (two-dimensional) closed curve and a double integral.

Definition

Let Ω be a domain in a plane bounded by a simple closed curve k. **The curve k is orientated positively** if traveling on the curve we always have got the domain Ω on the left side, see figure.



Remark

Positive orientation of the closed curve means traveling in a counterclockwise direction, while negative orientation means traveling in a clockwise direction.

We denote the line integral of a vector field $\mathbf{F}(X)$ along a closed curve k by

$$\oint_k \mathbf{F}(X) \cdot \,\mathrm{d}\mathbf{s}.$$

Theorem (Green's theorem)

Let two-dimensional vector field

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$$

have continuous partial derivatives on the plane domain Ω, which is bounded by simple piecewise smooth closed positively orientated curve k. Then

$$\oint_k P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \iint_\Omega \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right] \mathrm{d}x\,\mathrm{d}y.$$

The Green's theorem transforms the line integral of a vector field along a plane closed curve to a double integral over the domain Ω, that is bounded by the curve k. It is especially useful in situations when k is a polygon and we would have to calculate as many line integrals as the lines the polygon consists of.

Example 72

Calculate the integral $\oint_k (2xy - 5y)\,\mathrm{d}x + \left(x^2 + y\right)\,\mathrm{d}y$, where k is the positively orientated circle with the center in the origin of coordinates and radius r.

The curve k is simple and closed. Also both functions $P(x,y) = 2xy - 5y$ and $Q(x,y) = x^2 + y$ fulfil assumptions of the Green's theorem. We calculate derivatives

$$\begin{aligned} \frac{\partial P(x,y)}{\partial y} &= 2x - 5 \\ \frac{\partial Q(x,y)}{\partial x} &= 2x \end{aligned}$$

and evaluate the integral by using Green's theorem.

$$\oint_k (2xy - 5y)\ \mathrm{d}x + (x^2 + y)\ \mathrm{d}y = \iint_\Omega (2x - (2x - 5))\ \mathrm{d}x\,\mathrm{d}y = \iint_\Omega 5\ \mathrm{d}x\,\mathrm{d}y = 5\pi r^2.$$

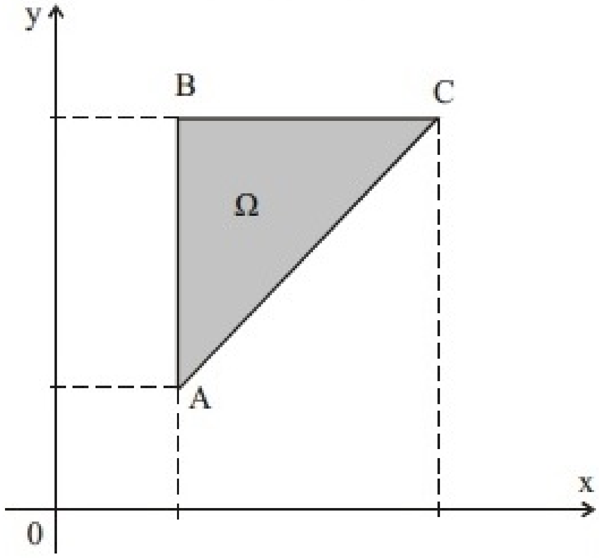
Example 73

Calculate the integral $\oint\limits_k \left(x^2+y^2\right)\,\mathrm{d}x + (x+y)^2\,\mathrm{d}y$, where k consists of the sides of the triangle ABC: $A=[1,1], B=[1,3], C=[3,3]$. The curve is positively orientated.

All assumptions of the Green's theorem are fulfilled.

$$P = x^2+y^2, \qquad Q=(x+y)^2,$$
$$\frac{\partial P(x,y)}{\partial y} = 2y, \qquad \frac{\partial Q(x,y)}{\partial x} = 2(x+y).$$

The domain is shown on following figure.



We calculate the double integral as a normal one with respect to the x-axis with inequalities for Ω in the form

$$\Omega:\quad 1\le x\le 3, \\ x\le y\le 3.$$

By using Green's theorem we obtain

$$\oint\limits_k \left(x^2+y^2\right)\,\mathrm{d}x + (x+y)^2\,\mathrm{d}y = \iint\limits_\Omega (2x+2y-2y)\,\mathrm{d}x\,\mathrm{d}y = 2\int\limits_1^3 \mathrm{d}x\int\limits_x^3 x\,\mathrm{d}y$$

$$= 2\int\limits_1^3 x[y]_x^3\,\mathrm{d}x = 2\int\limits_1^3 \left(3x-x^2\right)\,\mathrm{d}x = 2\left[\frac{3}{2}x^2-\frac{x^3}{3}\right]_1^3 = \frac{20}{3}.$$

Exercise 74

Calculate the line integrals of vector fields along given curves using Green's theorem.

a) $\oint\limits_k \left(x^2+y^2\right)\,\mathrm{d}y$, $\quad k$ consists of the sides of the rectangle $0\le x\le 2$, $0\le y\le 4$. The curve is positively orientated.

b) $\displaystyle\oint_k 2y\,\mathrm{d}x - (x+y)\,\mathrm{d}y,$ k consists of the sides of the triangle $x \geq 0, y \geq 0,$ $x + 2y \leq 4$. The curve is positively orientated.

c) $\displaystyle\oint_k (x+y)\,\mathrm{d}x - (x-y)\,\mathrm{d}y,$ k is positively orientated ellipse $4x^2 + 9y^2 = 36$.

d) $\displaystyle\oint_k \left(\mathrm{e}^x \sin y - 16y\right)\,\mathrm{d}x + \left(\mathrm{e}^x \cos y + 16\right)\,\mathrm{d}y$ k is positively orientated circle $x^2 + y^2 = 2x$.

4.3.2 Path independence of line integral

We can use another method of calculation of the line integral of a vector field in situations when the value of the integral doesn't depend on an integration curve and depends only on its starting and ending point.

Definition

Let points $A, B \in \Omega$. Let the vector function $\mathbf{F}(X)$ be continuous over the domain Ω. If the value of the line integral of the vector field

$$\int_k \mathbf{F}(X) \cdot \mathbf{ds}$$

doesn't depend on the integration curve k with starting point A and ending point B that lies within the domain Ω, we say **the integral is path independent between points** A, B. If this property is fulfilled for arbitrary points $A, B \in \Omega$, we say **integral is path independent over Ω.**

Theorem (Path independence of line integral)

Let $\mathbf{F}(X) = P(X)\mathbf{i} + Q(X)\mathbf{j} + R(X)\mathbf{k}$ have continuous partial derivatives in a domain Ω. Let a curve k lie within Ω, A be the starting point of the curve k, while B be its ending point. Then:

1. The line integral $\displaystyle\int_k \mathbf{F}(X) \cdot \mathbf{ds} = \int_k P(X)\,\mathrm{d}x + Q(X)\,\mathrm{d}y + R(X)\,\mathrm{d}z$ is path independent over Ω if and only if there exists some scalar function $\Phi(x,y,z)$ over Ω such that $F(x,y,z) = \mathbf{grad}\,\Phi(x,y,z)$, i.e.

$$P = \frac{\partial \Phi}{\partial x}, \qquad Q = \frac{\partial \Phi}{\partial y}, \qquad R = \frac{\partial \Phi}{\partial z}.$$

   The vector field $\mathbf{F}(x,y,z)$ is a **conservative field**, function $\Phi(x,y,z)$ is a **scalar potential**.

2. The line integral $\displaystyle\int_k \mathbf{F}(X) \cdot \mathbf{ds} = \int_k P(X)\,\mathrm{d}x + Q(X)\,\mathrm{d}y + R(X)\,\mathrm{d}z$ is path indepen-

dent over Ω if and only if $\mathbf{curl}\,\mathbf{F}(X) = \mathbf{o}$, i.e.

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

3. In such case, the line integral of the vector field $\mathbf{F}(X)$ along a curve k from the starting point A to the ending point B is given by the difference of the values of scalar potential in the end point B and starting point A :

$$\int_k \mathbf{F}(X) \cdot \mathbf{ds} = \int_k P(X)\,\mathrm{d}x + Q(X)\,\mathrm{d}y + R(X)\,\mathrm{d}z = \Phi(B) - \Phi(A).$$

4. Hence, if the line integral is path independent and the curve k is closed then

$$\oint_k \mathbf{F}(X) \cdot \mathbf{ds} = 0.$$

If the problem is considered only in xy-plane, the path independent line integral of the vector field is given by

$$\int_k \mathbf{F}(X) \cdot \mathbf{ds} = \int_k P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y = \Phi(B) - \Phi(A).$$

Two-dimensional vector field is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

We will show the way of calculation of the scalar potential $\Phi(x,y)$ at following examples.

Example 75

Calculate the integral $\int_k \left(3x^2 - 2xy + y^2\right)\,\mathrm{d}x - \left(x^2 - 2xy + 3y^2\right)\,\mathrm{d}y$, where k is oriented line segment $\overline{AB}$, $A = [1,2]$, $B = [3,1]$.

The test condition $\frac{\partial P}{\partial y} = -2x + 2y = \frac{\partial Q}{\partial x}$ is fulfilled.
To find the potential we first integrate $P(x,y) = 3x^2 - 2xy + y^2$ with respect to x.

$$\Phi(x,y) = \int P(x,y)\,\mathrm{d}x = \int \left(3x^2 - 2xy + y^2\right)\,\mathrm{d}x = x^3 - x^2y + xy^2 + K(y),$$

where $K(y)$ is a function of variable y. We determine it by setting the partial derivative $\frac{\partial \Phi}{\partial y}$ equal to $Q(x,y)$. We have

$$\frac{\partial \Phi}{\partial y} = -x^2 + 2xy + K'(y) = -x^2 + 2xy - 3y^2 = Q(x,y).$$

Hence, $K'(y) = -3y^2$ and

$$K(y) = -\int 3y^2\, dy = -y^3 + C,$$

where C is a real constant. The scalar potential is in the form

$$\Phi(x,y) = x^3 - x^2y + xy^2 - y^3 + C.$$

We calculate the integral according to path independence theorem:

$$\Phi(B) = 3^3 - 3^2 \cdot 1 + 3 \cdot 1^2 - 1^3 = 20, \qquad \Phi(A) = 1^3 - 1^2 \cdot 2 + 1 \cdot 2^2 - 2^3 = -5,$$

$$\int_k \left(3x^2 - 2xy + y^2\right)\, \mathrm{d}x - \left(x^2 - 2xy + 3y^2\right)\, \mathrm{d}y = \Phi(B) - \Phi(A) = 25.$$

Example 76

Calculate the integral $\oint_k x^2\, \mathrm{d}x + y^2\, \mathrm{d}y$ along the positively orientated closed curve $k\colon x^2 + y^2 = r^2$.

Functions $P = x^2$ and $Q = y^2$ fulfil assumptions of path independence theorem. Partial derivatives

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0.$$

Hence, the integral is path independent. The curve is closed. Therefore, the integral must be equal to zero.

$$\oint_k x^2\, \mathrm{d}x + y^2\, \mathrm{d}y = 0$$

Example 77

Calculate the integral $\int_k (2x + yz)\, \mathrm{d}x + \left(xz + z^2\right)\, \mathrm{d}y + (xy + 2yz)\, \mathrm{d}z$ from the starting point $A = [1,1,1]$ to the ending point $B = [1,2,3]$.

We determine functions P, Q, R and all needed partial derivatives:

$$P(x,y,z) = 2x + yz, \qquad \frac{\partial P(x,y,z)}{\partial y} = z, \qquad \frac{\partial P(x,y,z)}{\partial z} = y,$$

$$Q(x,y,z) = xz + z^2, \qquad \frac{\partial Q(x,y,z)}{\partial x} = z, \qquad \frac{\partial Q(x,y,z)}{\partial z} = x + 2z,$$

$$R(x,y,z) = xy + 2yz, \qquad \frac{\partial R(x,y,z)}{\partial x} = y, \qquad \frac{\partial R(x,y,z)}{\partial y} = x + 2z.$$

We can see that the test conditions are fulfilled

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \qquad \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}.$$

Hence, **curl F** $=$ **o** and integral is path independent.
To find the scalar potential Φ we use its properties from the definition of the scalar potential

$$P = \frac{\partial \Phi}{\partial x}, \quad Q = \frac{\partial \Phi}{\partial y}, \quad R = \frac{\partial \Phi}{\partial z}.$$

Hence,

$$\Phi = \int P\,\mathrm{d}x = \int (2x + yz)\,\mathrm{d}x = x^2 + xyz + K_1(y,z),$$

where $K_1(y,z)$ is an arbitrary function depending on variables y, z. We determine it by setting the partial derivative $\frac{\partial \Phi}{\partial y}$ equal to Q. We obtain

$$\frac{\partial \Phi}{\partial y} = xz + \frac{\partial K_1(y,z)}{\partial y} = xz + z^2 = Q$$

and

$$K_1(y,z) = \int z^2 dy = yz^2 + K_2(z),$$

where $K_2(z)$ is an arbitrary function depending only on variable z. We have

$$\Phi = x^2 + xyz + yz^2 + K_2(z).$$

We use equation $\frac{\partial \Phi}{\partial z} = R$ and we obtain

$$xy + 2yz + K_2'(z) = xy + 2yz.$$

Integrating this equation we get

$$K_2(z) = \int 0\,\mathrm{d}z = C,$$

where C is an arbitrary real constant. Finally, we obtained scalar potential in the form

$$\Phi(X) = x^2 + xyz + yz^2 + C.$$

We calculate the integral according to path independence theorem:

$$\Phi(B) = 1^2 + 1 \cdot 2 \cdot 3 + 2 \cdot 3^2 = 25, \qquad \Phi(A) = 1^2 + 1 \cdot 1 \cdot 1 + 1 \cdot 1^2 = 3,$$

$$\int_k (2x + yz)\,\mathrm{d}x + \left(xz + z^2\right)\,\mathrm{d}y + (xy + 2yz)\,\mathrm{d}z = \Phi(B) - \Phi(A) = 22.$$

Exercise 78

Prove that following integrals are path independent. Then, calculate them if A is the starting point and B is the ending point.

a) $\displaystyle\int_k \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2}, \quad A = [1,2],\ B = [2,3]$

b) $\displaystyle\int_k \left(2y - 6xy^3\right)\,\mathrm{d}x + \left(2x - 9x^2y^2\right)\,\mathrm{d}y, \quad A = [1,1],\ B = [4,1]$

c) $\displaystyle\int_k yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z, \quad A = [2,2,2],\ B = [2,3,4]$

d) $\displaystyle\int\limits_k \frac{\mathrm{d}x + 2\,\mathrm{d}y + 3\,\mathrm{d}z}{x + 2y + 3z}, \quad A = [0,1,0],\ B = [1,0,1]$

4.3.3 Practical applications of line integral of the vector field

Work in a force field

Suppose an object moving in a force field $\mathbf{F}$ along a curve k. Work done by the force $\mathbf{F}$ is then given by the line integral of a vector field

$$W = \int\limits_k \mathbf{F} \cdot \mathrm{d}\mathbf{s}.$$

Example 79

Calculate the work done by the force field $\mathbf{F} = (xy, x+y)$ on an object moving along the line segment $\overline{AB}$ from the point $A = [0,0]$ to the point $B = [1,1]$.

We describe the line segment by parametrization

$$\begin{aligned} x &= t, \\ y &= t, \quad t \in [0,1] \end{aligned}$$

The curve is positively oriented with respect to the parameter t. We calculate the derivatives $\dot{x} = 1$, $\dot{y} = 1$ and calculate the work by using line integral of a vector field

$$W = \int\limits_k \mathbf{F} \cdot \mathrm{d}\mathbf{s} = \int\limits_k xy\,\mathrm{d}x + (x+y)\,\mathrm{d}y$$

$$= \int\limits_0^1 \left(t^2 + 2t\right)\,\mathrm{d}t = \left[\frac{t^3}{3} + t^2\right]_0^1 = \frac{1}{3} + 1 = \frac{4}{3}.$$

Exercise 80

a) Calculate the work done by the force field $\mathbf{F} = (xy, x+y)$ on an object moving along the curve $k\colon x = y^2$ from the point $A = [0,0]$ to the point $B = [1,1]$.

b) Calculate the work done by the force field $\mathbf{F} = (x+y, 2x)$ on an object moving along the closed curve $k\colon x^2 + y^2 = r^2$ in a positive direction.

c) Calculate the work done by the force field $\mathbf{F} = \left(x^2, y^2, z^2\right)$ on an object moving around the screw line $k\colon x = \cos t$, $y = \sin t$, $z = t$, $t \in \left[0, \frac{\pi}{2}\right]$ in positive direction with respect to the parameter t.

d) Calculate the work done by the force field $\mathbf{F} = \mathbf{grad}(\Phi)$, $\Phi = \ln\left(x^2 + y^2\right) - \arctan\frac{x}{y}$ on an object moving from the point $A = [1,1]$ to the point $B = \left[\sqrt{2}, \sqrt{2}\right]$.

Name: Workbook for Mathematics III

Department, Institute: Faculty of Mechanical Engineering, Department of Mathematics and Descriptive Geometry

Department, Institute: Faculty of Civil Engineering, Department of Mathematics

Authors: Jakub Stryja, Arnošt Žídek

Place, year of publishing: Ostrava, 2024, 1st Edition

Number of Pages: 77

Published: VSB – Technical University of Ostrava

ISBN 978-80-248-4730-6 (on-line)