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MATHEMATICS I

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1. LINEAR ALGEBRA

1.1. Matrices

1.1.1. Definitions

***m x n* matrix with *m* rows and *n* columns is called an array of *m.n* real numbers**

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

The entry a_{ij} denotes **the element in the *i*th row and *j*th column**.

- If ***m = n*** then the array is square, and **A** is then called **square matrix of order *n***.
- In a square matrix of order *n*, the diagonal containing elements $a_{11}, a_{22}, \cdots, a_{nn}$ is called **principal** or **leading diagonal.**
- **Diagonal matrix D** is a square matrix that has its only non-zero elements along the leading diagonal.

$$\mathbf{D} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & & \cdots & a_{nn} \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & & \cdots & 1 \end{pmatrix}$$

- A very important special case of diagonal matrix is the **unit matrix** or **identity matrix E**, for which $a_{11} = a_{22} = ... = a_{nn} = 1$.
- The **zero** or **null matrix** is the matrix with every element zero.
- The **transposed matrix $\mathbf{A}^{\mathrm{T}}$** of matrix **A** is just the matrix with rows and columns interchanged.

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

- If the matrix has one row or one column

$$\underline{u} = (u_1, u_2, \cdots, u_n) \text{ or } \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = (v_1, v_2, \cdots, v_n)^T$$

then it is called a **row vector** or a **column vector.**

1.1.2. Basic Properties of Matrices

Equality A = B

Two matrices **A** and **B** are said to be **equal** if and only if all their corresponding elements are the same ($a_{ij} = b_{ij}$ for $\forall i, j$) and they are of the same order *m x n*.

Addition C = A + B

We can only add a *m x n* matrix to another *m* x *n* matrix, and an element of the sum is the sum of the corresponding elements.

$\mathbf{C} = (c_{ij}) = (a_{ij} + b_{ij})$ for $\forall i, j$.

Multiplication by a scalar k

The matrix **kA** has elements $\mathrm{ka_{ij}}$, i.e., we just multiply each element by the scalar k.

$\mathbf{D} = \mathbf{kA} = (d_{ij}) = (ka_{ij})$ for $\forall i, j$.

Matrix multiplication

If **A** is *m x p* matrix with elements $\mathrm{a_{ij}}$ and **B** a *p x n* matrix with elements $\mathrm{b_{ij}}$ then we define the **product C = A.B** as the *m x n* matrix with elements

$$\mathbf{C} = (c_{ij}) = (a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ip}b_{pj}) = \sum_{k=1}^{p} a_{ik}b_{kj} \text{, for } \mathrm{i} = 1, 2, \cdots, m \text{ and } \mathrm{j} = 1, 2, \cdots, n .$$

The *i*th row of **A** is multiplied term by term with the *j*th column of **B** to form the *ij*th component of **C**. In order for multiplication to be possible, **A** must have *p* columns and **B** must have *p* rows otherwise their product **A.B** is not defined.

Matrix multiplication is not commutative in general: **A.B** ≠ **B.A**.

Properties of the transpose:

From the definition, the transpose of matrix is such that

$$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}, \qquad (\mathbf{A.B})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}, \qquad (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A}.$$

Example 1.1.1: Given the matrices $\mathbf{A} = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 & -3 \\ 2 & -1 & 1 \end{pmatrix}$.

Find the matrices **A+B, A-B, B-A, 3A, 4B, 3A+4B,** $\mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$.

Solution: $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{A}-\mathbf{B} = \begin{pmatrix} 1 & 3 & 7 \\ -3 & 3 & -1 \end{pmatrix}$, $\mathbf{B}-\mathbf{A} = \begin{pmatrix} -1 & -3 & -7 \\ 3 & -3 & 1 \end{pmatrix}$,

$\mathbf{3A} = \begin{pmatrix} 6 & 9 & 12 \\ -3 & 6 & 0 \end{pmatrix}$, $\mathbf{4B} = \begin{pmatrix} 4 & 0 & -12 \\ 8 & -4 & 4 \end{pmatrix}$, $\mathbf{3A+4B} = \begin{pmatrix} 10 & 9 & 0 \\ 5 & 2 & 4 \end{pmatrix}$,

$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & -1 \\ 3 & 2 \\ 4 & 0 \end{pmatrix}$, $\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -3 & 1 \end{pmatrix}$, $\mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 3 & 1 \\ 3 & 1 \\ 1 & 1 \end{pmatrix}$.

Example1.1.2: Given the matrices $\mathbf{C} = \begin{pmatrix} 3 & 3 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ 2 & 0 \end{pmatrix}$.

Find the matrices **K = C.D** and **M = D.C**.

Solution: $\mathbf{K} = \mathbf{C.D} = \begin{pmatrix} 11 & 9 \\ 5 & 3 \end{pmatrix}$, $\mathbf{M} = \mathbf{D.C} = \begin{pmatrix} 9 & 9 & 5 \\ 3 & 3 & 1 \\ 6 & 6 & 2 \end{pmatrix}$,

k_{11} = 3.2+3.1+1.2 = 11,
k_{12} = 3.3+3.0+1.0 = 9,
k_{21} = 1.2+1.1+1.2 = 5,
k_{22} = 1.3+1.0+1.0 = 3.

m_{11} = 2.3+3.1 = 9, m_{12} = 2.3+3.1 = 9,
m_{13} = 2.1+3.1 = 5, m_{21} = 1.3+0.1 = 3,
m_{22} = 1.3+0.1 = 3, m_{23} = 1.1+0.1 = 1,
m_{31} = 2.3+0.1 = 6, m_{32} = 2.3+0.1 = 6,
m_{33} = 2.1+0.1 = 2.

$\mathbf{C.D} \neq \mathbf{D.C}$.

1.2. Determinants

1.2.1. Definition and Basic Properties

The determinant of order *n* > 1 a square matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

is defined as a number det $\mathbf{A}$ = $a_{11}D_{11} - a_{12}D_{12} + ... + (-1)^{1+n} a_{1n}D_{1n} =$

$$= a_{11}\cdot\begin{vmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \vdots \\ a_{n2} & \cdots & a_{nn} \end{vmatrix} - a_{12}\cdot\begin{vmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} + ... + (-1)^{1+n} a_{1n}\cdot\begin{vmatrix} a_{21} & \cdots & a_{2,n-1} \\ \vdots & \cdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} \end{vmatrix}.$$

We write det $\mathbf{A}$ = A = $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

The commonly useful properties are as follows:

- Thus two rows (or columns) are the same, the determinant is zero.
- We multiply a determinant by a number c if we multiply by this number c all the elements of a row or of a column.
- Interchanging two rows (or columns) changes the sign of the determinant.
- The addition rule:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & ... & b_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Similarly for the columns.

• Adding multiples of rows (or columns):

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + c \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}+ca_{21} & a_{12}+ca_{22} & \dots & a_{1n}+ca_{2n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

1.2.2. Evaluation of Determinants

For $n = 1$, is det **A** = a_{11},

for $n = 2$, is det **A** = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$.

Example 1.2.1: Evaluate the determinant det **B** = $\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix}$.

Solution: det **B** = 1.5 – 3.4 = 5 - 12 = -7

Sarrus´s rule for a determinant of the third order:

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}.$$

Example 1.2.2: Evaluate the determinant det **C** = $\begin{vmatrix} 6 & 1 & 2 \\ 0 & 3 & -1 \\ 4 & 2 & 1 \end{vmatrix}$.

Solution: det **C**= 6.3.1 + 0.2.2 + 4.1.(-1) – [2.3.4 + (-1).2.6 + 1.1.0] =

= 18 + 0 - 4 - (24 - 12 + 0) = 14 - 12 = 2

In general for the determinant of order $n > 1$

the **Laplace´s expansion according to the *i*th row** holds:

$$\det \mathbf{A} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} A_{ij} = (-1)^{i+1} a_{i1} A_{i1} + (-1)^{i+2} a_{i2} A_{i2} + \dots + (-1)^{i+n} a_{in} A_{in},$$

or the **Laplace´s expansion according to the *j*th column** holds:

$$\det \mathbf{A} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij} = (-1)^{1+j} a_{1j} A_{1j} + (-1)^{2+j} a_{2j} A_{2j} + \dots + (-1)^{n+j} a_{nj} A_{nj} .$$

Note that

- $(-1)^{i+j}$ is called a **sign of element** a_{ij} ,

- **determinant** A_{ij} originating from det **A** omitting the *i*th row and *j*th column, is called

  minor of order *n-1* of det A belonging to the element a_{ij} **,**

- **cofactor** A^*_{ij} **of the element** $\mathbf{a_{ij}}$ is defined as the minor A_{ij} multiplied by the appropriate

  sign $(-1)^{i+j}$: $A^*_{ij} = (-1)^{i+j}\ A_{ij}$.

Example 1.2.3: Evaluate the determinant det **D** = $\begin{vmatrix} 2 & 0 & 4 & 2 \\ 3 & -2 & 3 & 1 \\ 0 & 4 & 5 & -1 \\ -1 & 2 & 1 & 3 \end{vmatrix}$

Solution: The second column contains only even numbers therefore we can put it in form of product 2*det D_a.

We do the Laplace´s expansion according to the second column in the second step.

$$\det \mathbf{D} = 2.\begin{vmatrix} 2 & 0 & 4 & 2 \\ 3 & -1 & 3 & 1 \\ 0 & 2 & 5 & -1 \\ -1 & 1 & 1 & 3 \end{vmatrix} = 2.\begin{vmatrix} 2 & 0 & 4 & 2 \\ 3 & -1 & 3 & 1 \\ 6 & 0 & 11 & 1 \\ 2 & 0 & 4 & 4 \end{vmatrix} = 2\ (-1)(-1)^{2+2}\begin{vmatrix} 2 & 4 & 2 \\ 6 & 11 & 1 \\ 2 & 4 & 4 \end{vmatrix} = \text{(Sarrus´s rule)} =$$

$$= -2\ [88 + 48 + 8 - (\ 44 + 8 + 96\ )] = 8 \text{ (Sarrus´s rule)}$$

or Laplace´s expansion according to the third row:

$$\det \mathbf{D} = -2\begin{vmatrix} 2 & 4 & 2 \\ 6 & 11 & 1 \\ 2 & 4 & 4 \end{vmatrix} = -2\begin{vmatrix} 2 & 4 & 2 \\ 6 & 11 & 1 \\ 0 & 0 & 2 \end{vmatrix} = -2.2.(-1)^{3+3}\begin{vmatrix} 2 & 4 \\ 6 & 11 \end{vmatrix} = 8\ .$$

1.2.3. Matrix Inversion

The determinant of order *k* formed by the elements in the intersections of arbitrary *k* rows and *k* columns of matrix **A** = $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$

is called a **minor of order *k* of the matrix A** ($1 \le k \le \min(m, n)$).

A **matrix A is of rank h** if and only if there exists a minor of *A* of order **h** different from zero, any minor of *A* of order higher than **h** being equal to zero.

The square matrix A is called

- **regular**, if det $\mathbf{A} \neq \mathbf{0}$,
- **singular**, if det $\mathbf{A} = \mathbf{0}$.

The inverse of the **square matrix A** of order *n* is a **square matrix** $\mathbf{A^{-1}}$ of order *n* such that

$$\mathbf{A.A^{-1}} = \mathbf{A^{-1}.A = E}\ ,$$

where **E** is unit matrix.

The inverse matrix $\mathbf{A^{-1}}$ of the square matrix **A** exist if and only if **A** is regular.

Then holds $\mathbf{A^{-1}} = \dfrac{\mathbf{1}}{\mathbf{det\ A}}\mathbf{adj\ A}$,

Note that

• **adjoint matrix adj A** is defined as the transpose of matrix of cofactors A^*_{ij}, that is

$$\mathbf{adj\ A} = \begin{pmatrix} A^*_{11} & A^*_{12} & \cdots & A^*_{1n} \\ A^*_{21} & A^*_{22} & \ldots & A^*_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A^*_{n1} & A^*_{n2} & \ldots & A^*_{nn} \end{pmatrix}^T = \begin{pmatrix} A^*_{11} & A^*_{21} & \cdots & A^*_{n1} \\ A^*_{12} & A^*_{22} & \ldots & A^*_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ A^*_{1n} & A^*_{2n} & \ldots & A^*_{nn} \end{pmatrix}$$

Example 1.2.4: Find the inverse matrix $\mathbf{A^{-1}}$ of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$.

Solution: Determinant of matrix **A**: det **A** = $\begin{vmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{vmatrix} = -2+6+0-3+0-2=-1$,

the matrix of cofactors $\mathrm{A}^*_{\mathrm{ij}}$:

$$\mathrm{A}^*_{\mathrm{ij}} = \begin{pmatrix} (-1)^{1+1}\begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} & (-1)^{1+2}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} & (-1)^{1+3}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \\ (-1)^{2+1}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} & (-1)^{2+2}\begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} & (-1)^{2+3}\begin{pmatrix} 2 & 2 \\ -1 & 2 \end{pmatrix} \\ (-1)^{3+1}\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix} & (-1)^{3+2}\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} & (-1)^{3+3}\begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 4 & 5 & -6 \\ 3 & 3 & -4 \end{pmatrix},$$

the adjoint matrix **adj A =** $\begin{pmatrix} -1 & -1 & 1 \\ 4 & 5 & -6 \\ 3 & 3 & -4 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & -6 & -4 \end{pmatrix}$,

$$\mathbf{A^{-1}} = \frac{1}{\det \mathbf{A}}\mathbf{adj\ A} = \frac{1}{-1}\begin{pmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & -6 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}.$$

Test: $\mathbf{A.\,A^{-1}} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & 1 \end{pmatrix} . \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

1.2. Systems of Linear Equations

1.2.3. Definition

By a **system of *m* linear equations in *n* unknowns** we understand the system

$$\boxed{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}},$$

where

- $x_1, x_2, \ldots, x_n$ are called **unknowns**,
- real numbers a_{ij} ($i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$) are called **coefficients**,
- real numbers b_i are called **right-hand side**,
- if real numbers $b_i = 0$ ($i = 1, 2, \ldots, m$) system of linear equations **is** called **homogeneouse**,

  if exists $b_i \neq 0$, system of linear equations is called **nonhomogeneouse.**
- matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is called **matrix of the system**,
- matrix $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is called **solution vector**,
- matrix $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ is called **right-hand side vector**,
- matrix $\mathbf{A|B} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$ is called **augmented matrix of the system.**

Then a system of equations can be rewritten in the matrix form **A.X = B** .

Theorem of Frobenius:

The system of *m* linear equations in *n* unknowns is solvable if and only if rank of matrix of the system is equal to rank of the augmented matrix of the system: h(A) = h(A|B) = h.

If **h = *n***, then system has only **one solution** ,

If **h < *n***, then system has ***n*-h linearly independent solutions** and every solution of this system is a linear combination of this *n*-h solutions.

The homogeneouse system of *m* linear equations in *n* unknowns A.X = O

a) has always the **trivial (zero) solution X** = $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

b) has a **non-trivial solution** if and only if det **A = 0**.

1.3.2. Gaussian Elimination

Given system of n linear equations in n unknowns $x_1, x_2, \ldots, x_n$ we solve in a serie of steps:

1. The essence of the Gaussian elimination consists in transforming the system of m linear equations in n unknowns into equivalent system (possesses the same solution) whose augmented matrix is upper triangular, i.e. the matrix $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & a_{mn} & b_m \end{pmatrix}$.

Elimination procedures rely on the manipulation of equations or, equivalently, the rows of the augmented matrix of the system. There are various **elementary row operations** used which do not alter the solution of the equations:

- multiply a row by a constant,
- interchange any two row,
- add or subtract one row from another.

2. We solve the last equation for x_n.

3. We then solve the penultimate equation and eliminate x_{n-1} in terms of x_n.

3. We repeat the process in turn $x_{n-2}, x_{n-3}, \ldots, x_2$ until we arrive at a final equation for x_1, which we can then solve.

Example 1.3.1: Solve the system of equations: $\begin{matrix} x_1 & + & 2x_2 & + & 5x_3 & = & -9 \\ x_1 & - & x_2 & + & 3x_3 & = & 2 \\ 3x_1 & - & 6x_2 & - & x_3 & = & 25 \end{matrix}$.

Solution: $m = n = 3$

x_1	x_2	x_3	**b**	**Σ**	**operation**
1	2	5	-9	-1	
1	-1	3	2	5	r_2-r_1
3	-6	-1	25	21	r_3-3r_1
1	2	5	-9	-1	
0	-3	-2	11	6	
0	-12	-16	52	24	r_3-4r_2
1	2	5	-9	-1	
0	-3	-2	11	6	
0	0	-8	8	0	

h(**A**) = **h**(**A**|**B**) = n = 3,

equivalent upper triangular augmented matrix of the system:

$$\begin{array}{llllllll}
(1) & x_1 & + & 2x_2 & + & 5x_3 & = & -9 \\
(2) & & - & 3x_2 & - & 2x_3 & = & 11 \\
(3) & & & & - & 8x_3 & = & 8
\end{array} \Rightarrow \text{we solve } x_3 = -1,$$

we solve then (2): $x_2 = -\dfrac{11+2x_3}{3} = -\dfrac{11+2(-1)}{3} = -3$,

and we solve finally (1): $x_1 = -9 - 2x_2 - 5x_3 = -9 + 6 + 5 = 2$

Example 1.3.2: Solve the system of equations
$$\begin{array}{rcrcrcr}
x_1 & + & x_2 & + & x_3 & = & 3 \\
x_1 & + & x_2 & - & 3x_3 & = & -1 \\
x_1 & + & 2x_2 & - & 3x_3 & = & 1 \\
2x_1 & + & x_2 & - & 2x_3 & = & 1
\end{array}.$$

Solution: $m = 4$, $n = 3$.

x_1	x_2	x_3	**b**	**Σ**	**operation**
1	1	1	3	6	
1	1	-3	-1	-2	r_2-r_1
1	2	-3	1	1	r_3-r_1
2	1	-2	1	2	r_4-$2r_1$
1	1	1	3	6	
0	0	-4	-4	-8	$r_2 \leftrightarrow r_3$
0	1	-4	-2	-5	
0	-1	-4	-5	-10	
1	1	1	3	6	
0	1	-4	-2	-5	
0	0	-4	-4	-8	
0	-1	-4	-5	-10	r_4+r_2
1	1	1	3	6	
0	1	-4	-2	-5	
0	0	-4	-4	-8	
0	0	-8	-7	-15	r_4-$2r_3$
1	1	1	3	6	
0	1	-4	-2	-5	
0	0	-4	-4	-8	
0	0	0	1	1	

$\mathbf{h}(\mathbf{A}) = 3$, $\mathbf{h}(\mathbf{A}|\mathbf{B}) = 4 \Rightarrow \mathbf{h}(\mathbf{A}) \neq \mathbf{h}(\mathbf{A}|\mathbf{B}) \Rightarrow$ system has no solution

(last row: $0\,x_1 + 0\,x_2 + 0\,x_3 = 1$ is not true.)

Example 1.3.3: Solve the system of equations:
$$\begin{array}{rcrcrcrcr}
4x_1 & + & x_2 & - & 3x_3 & - & x_4 & = & 0 \\
2x_1 & + & 3x_2 & + & x_3 & - & 5x_4 & = & 0 \\
x_1 & - & 2x_2 & - & 2x_3 & + & 2x_4 & = & 0
\end{array}.$$

Solution: homogeneous systém with $m = 3$, $n = 4$.

x_1	x_2	x_3	x_4	b	Σ	operation
1	-2	-2	2	0	-1	
2	3	1	-5	0	1	r_2-2r_1
4	1	-3	-1	0	1	r_3-4r_1
1	-2	-2	2	0	-1	
0	7	5	-9	0	3	r_2.9
0	9	5	-9	0	5	r_3·(-7)
1	-2	-2	2	0	-1	
0	63	45	-81	0	27	r_2:9
0	-63	-35	63	0	-35	r_3+r_2
1	-2	-2	2	0	-1	
0	7	5	-9	0	3	
0	0	10	-18	0	-8	

h(**A**) = **h**(**A**|**B**) = 3, n = 4 $\Rightarrow$ ***n*-h = 4 – 3 = 1 linearly independent solution** $x_4 = p$.

equivalent upper triangular augmented matrix of the system:

$$\begin{array}{lccccccccc}
(1) & x_1 & - & 2x_2 & - & 2x_3 & + & 2x_4 & = & 0 \\
(2) & & & 7x_2 & + & 5x_3 & - & 9x_4 & = & 0 \\
(3) & & & & & 10x_3 & - & 18x_4 & = & 0
\end{array}$$

(3) $10x_3 - 18p = 0 \;\Rightarrow\; x_3 = \dfrac{9p}{5}$,

(2) $7x_2 + 5x_3 - 9p = 0 \quad \Rightarrow \quad x_2 = \dfrac{9p\ \ -5x_3}{7} = \dfrac{9p\ \ -9p}{7} = 0$,

(1) $x_1 - 2x_2 - 2x_3 + 2p = 0 \quad \Rightarrow$

$$x_1 = -2p + 2x_2 + 2x_3 = -2\mathrm{p}\ + \frac{18p}{5} + 0 = \frac{8p}{5} .$$

For example: $p = 0:\ x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0$ (trivial solution),
$p = 5:\ x_1 = 8,\ x_2 = 0,\ x_3 = 9,\ x_4 = 5$,
$p = -5: x_1 = -8, x_2 = 0, x_3 = -9, x_4 = -5$

1.3.3. Cramer's Rule

The system of n linear equations in n unknowns $\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{n}$

$$\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & & + & a_{2n}x_n & = & b_2 \\
\vdots & & & & & & & & \\
a_{n1}x_1 & + & a_{n2}x_2 & + & \dots & + & a_{nn}x_n & = & b_n
\end{array}$$

with a regular matrix of the system has a unique solution $x_1, x_2, \dots, x_n$,

where $$x_i = \frac{\det \mathbf{D}_i}{\det \mathbf{D}},\ \ i = 1, 2, \dots, n\ \ ,$$

here det **D** is the determinant of the matrix of the system and $\det \mathbf{D}_i$ is the determinant obtaining by replacing the *i*th column of det D by the column of elements forming the right-hand side of equations.

Example1.3.4: Solve the system of equations:
$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & + & 2x_3 & = & 5 \\ x_1 & - & 5x_2 & + & 2x_3 & = & 7 \\ & & 7x_2 & + & 3x_3 & = & -7 \end{array}$$

Solution: $m = n = 3$.

$$\det \mathbf{D} = \begin{vmatrix} 3 & 1 & 2 \\ 1 & -5 & 2 \\ 0 & 7 & 3 \end{vmatrix} = -76 \ (\neq 0).$$

$$\det \mathbf{D}_1 = \begin{vmatrix} 5 & 1 & 2 \\ 7 & -5 & 2 \\ -7 & 7 & 3 \end{vmatrix} = -152, \qquad \det \mathbf{D}_2 = \begin{vmatrix} 3 & 5 & 2 \\ 1 & 7 & 2 \\ 0 & -7 & 3 \end{vmatrix} = 76, \qquad \det \mathbf{D}_3 = \begin{vmatrix} 3 & 1 & 5 \\ 1 & -5 & 7 \\ 0 & 7 & -7 \end{vmatrix} = 0,$$

$$x_1 = \frac{\det \mathbf{D}_1}{\det \mathbf{D}} = \frac{-152}{-76} = 2, \qquad x_2 = \frac{\det \mathbf{D}_2}{\det \mathbf{D}} = \frac{76}{-76} = -1, \qquad x_3 = \frac{\det \mathbf{D}_3}{\det \mathbf{D}} = \frac{0}{-76} = 0 \ .$$

2. DIFFERENTIAL CALCULUS FUNCTIONS OF ONE REAL VARIABLE

2.1. Functions of One Real Variable

2.1.1. Definitions and Basic Properties

A real-value function *f* relates each element x of a set $D(f)$, with exactly one element *y* of another set $H(f)$. We express the relationship by the equation $y = f(x)$ or *f: x*$\to$*y*.

- $D(f)$ is the **domain of f** and $H(f)$ is the **range of f** or **codomain.**
- Symbol ***x*** is an **independent variable** or **argument** of the function and the symbol ***y*** is the **dependent variable**.
- The g**raph** of function is the set of ordered pairs $[x, f(x)]$ for $\forall x \in D(f)$.

Bounded functions

We say that the function f is **bounded above** on a set $M \subseteq D(f)$ if there exists a number *h* such that $f(x) \le h \quad \forall x \in M$ (Fig. 1a),

we say that the function f is **bounded below** on a set $M \subseteq D(f)$ if there exists a number *d* such that $f(x) \ge d \quad \forall x \in M$ (Fig. 1b),

the function f is **bounded** provided it is bounded both from below and from above, it is if there exists a number *k* for which $|f(x)| \le k \quad \forall x \in M$ (Fig. 1c).

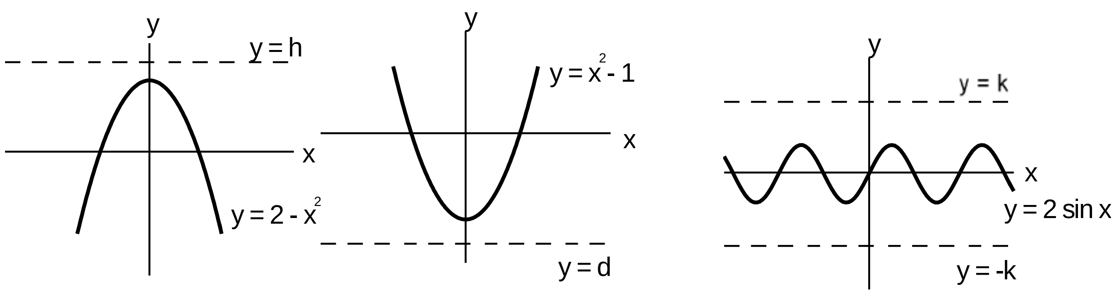

Fig. 1a Fig. 1b Fig. 1c

Monotonic functions

We say that the function *f* is **increasing** on an interval $I \subseteq D(f)$ provided that for $\forall x_1, x_2 \in I$ **:** $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ (Fig. 2a),

we say that the function f is **decreasing** on an interval $I \subseteq D(f)$ provided that for $\forall x_1, x_2 \in I$ **:** $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ (Fig. 2b),

we say that the function f is **nondecreasing** on an interval $I \subseteq D(f)$ provided that for $\forall x_1, x_2 \in I$ **:** $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ (Fig. 2c),

We say that the function f is **nonincreasing** on an interval $I \subseteq D(f)$ provided that for $\forall x_1, x_2 \in I$ **:** $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ (Fig. 2d).

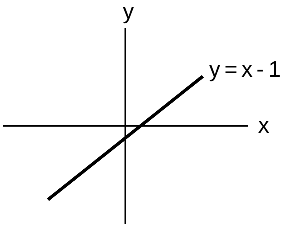

Fig. 2a

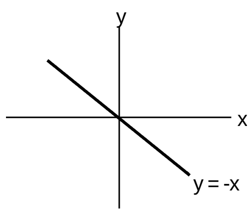

Fig. 2b

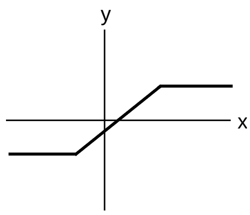

Fig. 2c

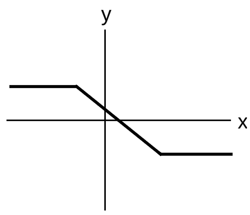

Fig. 2d

Periodic function

We say that the function f is **periodic** with period T on $D(f)$ provided that
for $\forall x \in D(f)$: $\quad f(x+T) = f(x) \quad$ (Fig. 3).

One-to-one function

We say that the function f is **one-to-one** on a set $M \subseteq D(f)$ provided that
for $\forall x_1, x_2 \in M$ **:** $\quad x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \quad$ (Fig. 4).

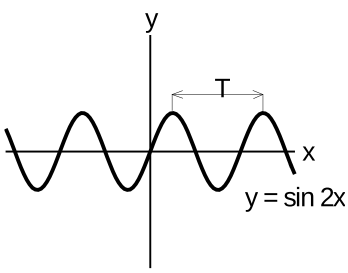

Fig. 3

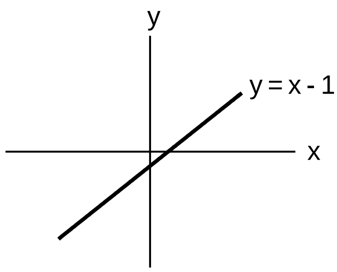

Fig. 4ˇ

Composite function

Let two functions $y = f(u)$ and $u = g(x)$ be given such that the domain of $f(u)$ intersects with the range of $g(x)$. Then the **composite** function $y = h(x)$ is defined to be the function *h*, for which domain *D* consists of all $x \in D(g)$) such that $g(x)$ lie in $D(f)$ and $h(x) = f(g(x))$ for every $x \in D(h)$, (Fig. 5).

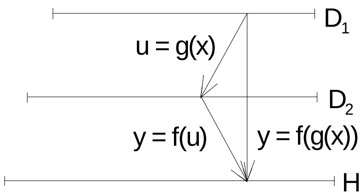

Fig. 5

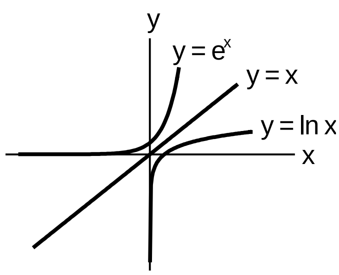

Fig. 6

Inverse functions

Let be an one-to-one function *f* with domain $D(f)$ and range $H(f)$, i.e. for every $y \in H(f)$ there exists $x \in D(f)$ x∈I for which $y = f(x)$. Then we define **inverse** function of *f* with domain $H(f)$ and range $D(f)$ by $x = f^{-1}(y)$ if and only if $f(x) = y$.

Properties of inverse function:

- Graph of the function f^{-1} is the reflection of the graph of f in the line $y = x$ (Fig. 6).
- Composition of inverse functions:

$f^{-1}(f(y)) = y$ for $\forall y \in H(f)$, $f^{-1}(f(x)) = x$ for $\forall x \in D(f)$.

- For domain and range of the inverse functions is valid:

$D(f) = H(f^{-1})$, $H(f) = D(f^{-1})$.

Even and odd functions

We say that the function f is **even** on an interval $(a,-a) \subseteq D(f)$ provided that
for $\forall x \in (a,-a)$: $f(-x) = f(x)$.

Graph of the even functions is symmetrical about vertical axis (Fig. 7a).

We say that the function f is **odd** on an interval $(a,-a) \subseteq D(f)$ provided that
for $\forall x \in (a,-a)$: $f(-x) = -f(x)$.

Graph of the odd functions is symmetrical about the origin (Fig.7b).

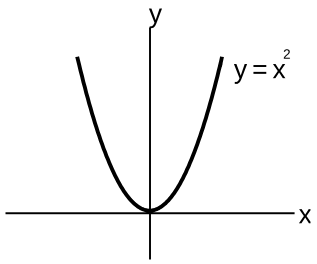

Fig. 7a

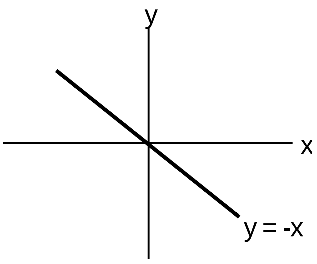

Fig. 7b

2.1.2. Elementary Functions

Polynomial functions

A polynomial function has the general form $\boxed{f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}$, where n is a positive integer and coefficients a_i, i = 0, 1, 2, …, n, $a_n \neq 0$ are real numbers. The index n is called the degree of the polynomial. Domain $D(f) = R = (-\infty, +\infty)$.

***n* = 0: constant function** (polynomial of degree 0): $\boxed{y = a_0 = k}$

Graph is a line parallel to x-axis, constant function is not one-to-one function (Fig. 8),

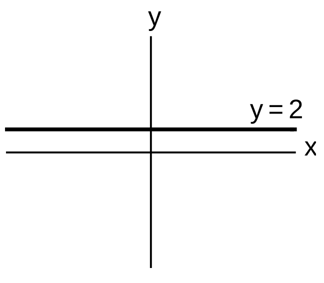

Fig. 8

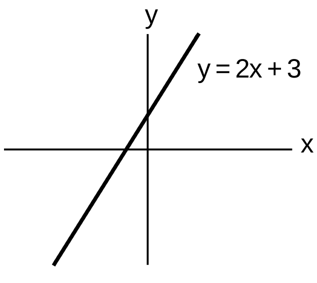

Fig. 9a

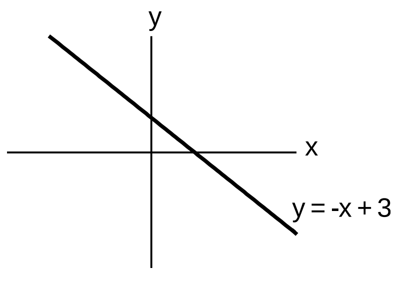

Fig. 9b

n = 1: **linear function** (polynomial of degree 1): $\boxed{y = a_1 x + a_0 = kx + q}$

Graph is a line with slope $k = tg\alpha$ which intersects y-axis at [0, q].

For $k > 0$ is linear function increasing (Fig. 9a), for $k < 0$ is decreasing (Fig. 9b), linear function is one-to-one.

***n* = 2: quadratic function** (polynomial of degree 2): $\boxed{y = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c}$

Graph is a parabola, it is not one-to-one function (Fig. 10a, b).

For $a > 0$ it is bounded below (Fig. 10a), for $a < 0$ it is bounded above (Fig. 10b).

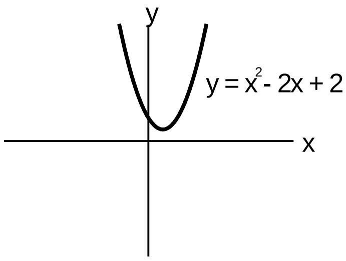

Fig. 10a

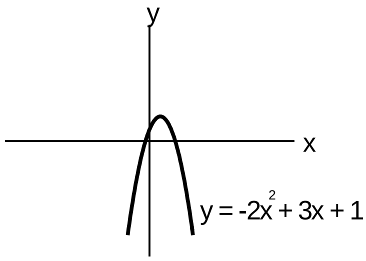

Fig. 10b

Rational function

A rational function has the general form $\boxed{y = \dfrac{P_m(x)}{Q_n(x)}}$,

where $P_m(x)$ and $Q_n(x)$ are polynomials of degree m a n.

Domain $D(f) = R - \{x \in R : Q(x) = 0\}$.

If $m < n$ it is said to be a proper rational function, if $m \geq n$ it is improper rational function. An improper rational function can always be expressed as a polynomial function plus a proper rational function by algebraic division.

Exponential function

An exponential function has the general form $\boxed{y = a^x}$,

where real constant a is called base, $a > 0$, $a \neq 1$.

Domain $D(f) = R = (-\infty, +\infty)$, range $H(f) = R_+ = (0, +\infty)$.

Graph of the exponential function is called *exponential curve* (Fig. 11a, b, c).

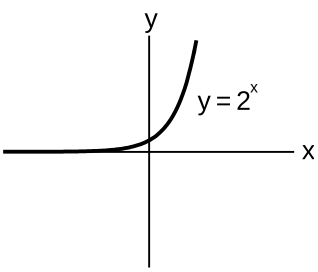

Fig. 11a

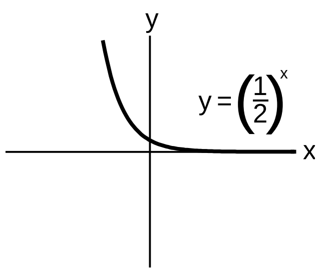

Fig. 11b

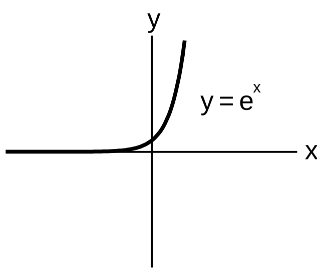

Fig. 11c

It is bounded below and it is increasing for $a > 1$ (Fig. 11a) and it is decreasing for $0 < a < 1$ (Fig. 11b), it is one-to-one function and it intersects y-axis at [0, 1].

The standard exponential function is $y = e^x$, (Fig. 11c), **Euler's number e = 2,71828**... .

Let us recall some basic properties:

$$a^r . a^s = a^{r+s}, \qquad \frac{a^r}{a^s} = a^{r-s}, \qquad (a^r)^s = a^{rs}, \qquad a^0 = 1.$$

Logarithmic functions

A logarithmic function has the general form $\boxed{y = \log_a x}$,

where real constant a is called base, $a > 0$, $a \neq 1$.

It is defined as inverse function of exponential function $y = a^x$, that is:

Domain $D(f) = R_+ = (0, +\infty)$, range $H(f) = R = (-\infty, +\infty)$.

Graph of the logarithmic function is called *logarithmic curve* (Fig. 12a, b, c).

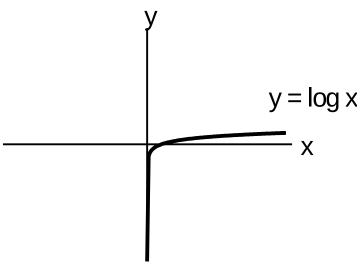

Fig. 12a

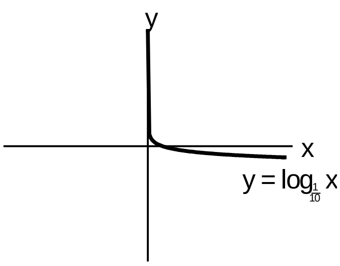

Fig. 12b

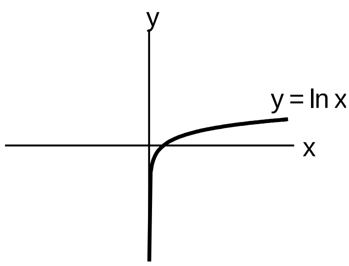

Fig. 12c

For $a > 1$ it is increasing (Fig. 12a), for $0 < a < 1$ it is decreasing (Fig. 12b),

it is one-to-one function and it intersects x-axis at [1, 0].

The inverse function of standard exponential function y = $y = e^x$ is called the natural logarithmic function and is written $y = \ln x$ (Fig. 12c).

Let us recall some basic properties:

for $u > 0$, $v > 0$ is valid: $\log_a u.v = \log_a u + \log_a v$, $\log_a a = 1$ $(a^1 = a)$,

$\log_a \dfrac{u}{v} = \log_a u - \log_a v$, $\log_a 1 = 0$ $(a^0 = 1)$, $\log_a u^v = v \log_a u$.

Trigonometric (circular) functions

All trigonometric functions are periodic, hence trigonometric functions are not one-to-one.

Functions $\boxed{y = \sin x}$ and $\boxed{y = \cos x}$ have basic properties:

Domain $D(f) = R = (-\infty, +\infty)$, range $H(f) = \langle -1, 1 \rangle$, hence they are bounded (Fig. 13a, b).

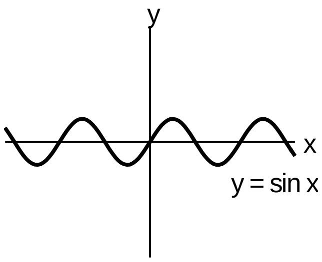

Fig. 13a

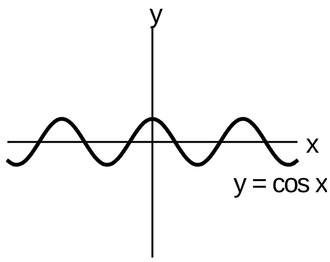

Fig. 13b

Functions are periodic with period $T = 2\pi$:

$\sin(x + 2k\pi) = \sin x$, $\cos(x + 2k\pi) = \cos x$, k is integer number.

Function $y = \sin x$ is odd function: $\sin(-x) = -\sin x$,

Function $y = \cos x$ is even function: $\cos(-x) = \cos x$.

Functions $\boxed{y = \operatorname{tg} x = \dfrac{\sin x}{\cos x}}$ and $\boxed{y = \operatorname{cotg} x = \dfrac{\cos x}{\sin x}}$ have basic properties:

Domain $D(\operatorname{tg} x) = R - \{(2k+1)\dfrac{\pi}{2}\}$, $D(\operatorname{cotg} x) = R - \{k\pi\}$, k is integer number,

range $H(\operatorname{tg} x) = H(\operatorname{cotg} x) = R = (-\infty, +\infty)$, hence they are not bounded (Fig. 14a, b).

Functions are periodic with period $T = \pi$:

$\operatorname{tg}(x + k\pi) = \operatorname{tg} x$, $\operatorname{cotg}(x + k\pi) = \operatorname{cotg} x$, k is integer number.

Functions $y = \operatorname{tg} x$ and $y = \operatorname{cotg} x$ are odd functions:

$\operatorname{tg}(-x) = -\operatorname{tg} x$, $\operatorname{cotg}(-x) = -\operatorname{cotg} x$.

Function $y = \operatorname{tg} x$ is increasing on the intervals $\left((2k-1)\dfrac{\pi}{2}, (2k+1)\dfrac{\pi}{2}\right)$.

Function $y = \operatorname{cotg} x$ is decreasing on the intervals $(k\pi, (k+1)\pi)$

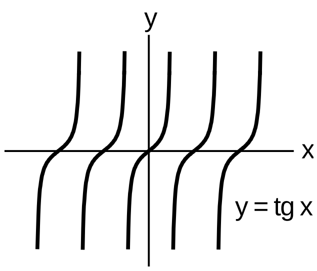

Fig. 14a

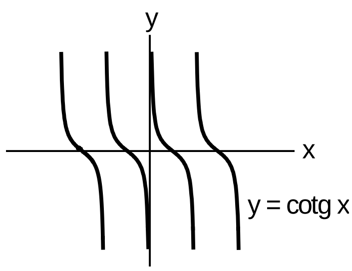

Fig. 14b

Let us recall some basic properties:

$\sin^2 x + \cos^2 x = 1$, $\operatorname{tg} x \,.\, \operatorname{cotg} x = 1$,

$\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x$,

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Inverse trigonometric functions

None of the trigonometric functions are one-to-one since they are periodic. In order to define inverses, it is customary to restrict the domains in which the trigonometric functions are one-to-one as follows.

Function $y = \sin x$ is increasing on the interval $< -\frac{\pi}{2}, \frac{\pi}{2} >$, hence it is one-to-one on this interval and it covers the range <-1,1>. So its inverse function exists and is denoted by $\boxed{y = \arcsin x}$.

We define $y = \arcsin x$, $x \in <-1,1>$, if and only if, $x = \sin y$, $y \in < -\frac{\pi}{2}, \frac{\pi}{2} >$.

Domain $D(\arcsin x)$ = $<-1,1>$, range $H(\arcsin x)$ = $< -\frac{\pi}{2}, \frac{\pi}{2} >$.

Function $y = \arcsin x$ is increasing on its domain (Fig. 15a).

Function $y = \cos x$ is decreasing on the interval $< 0, \pi >$, hence it is one-to-one on this interval and covers the range <-1, 1>. So its inverse function exists and is denoted by $\boxed{y = \arccos x}$.

We define $\arccos x$ $y = \arccos x$, $x \in <-1,1>$, if and only if, $x = \cos y$, $y \in < 0, \pi >$.

Domain $D(\arccos x)$ = $<-1,1>$, range $H(\arccos x)$ = $< 0, \pi >$.

Function y=arccos x is decreasing on its domain (Fig. 15b).

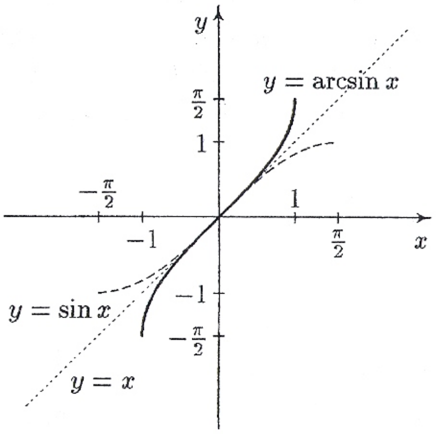

Fig. 15a

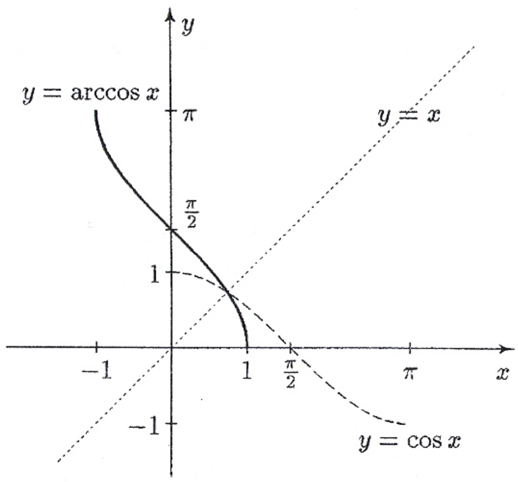

Fig. 15b

Function $y = \operatorname{tg} x$ is increasing on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, hence it is one-to-one on this interval and it covers the range (-∞, +∞). So its inverse function exists and is denoted by

$$\boxed{y = \arctan x}$$

We define $y = \arctan x$, $x \in (-\infty, +\infty)$, if and only if, $x = \operatorname{tg} y$, $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Domain $D(\arctan x) = (-\infty, +\infty)$, range $H(\arctan x) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Function $y = \arctan x$ is increasing on its domain (Fig. 16a).

Function $y = \operatorname{cotg} x$ is decreasing on the interval $(0, \pi)$, hence it is one-to-one on this interval and covers the range (-∞, +∞). So its inverse function exists and is denoted by

$$\boxed{y = \operatorname{arc cot} x}$$

We define $y = \operatorname{arc cot} x$, $x \in (-\infty, +\infty)$. if and only if, $x = \operatorname{cotg} y$, $y \in (0, \pi)$.

Domain $D(\operatorname{arc cot} x) = (-\infty, +\infty)$. range $H(\operatorname{arc cot} x) = (0, \pi)$.

Function $y = \operatorname{arc cot} x$ is decreasing on its domain (Fig. 16b).

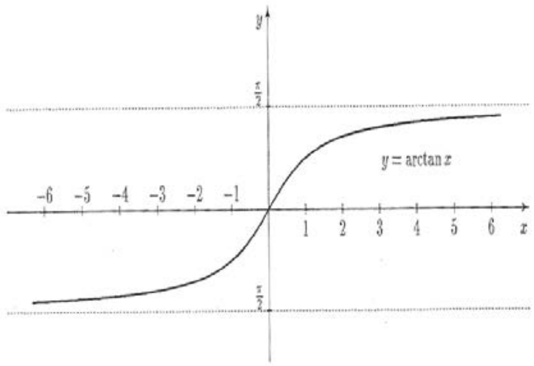

Fig. 16a

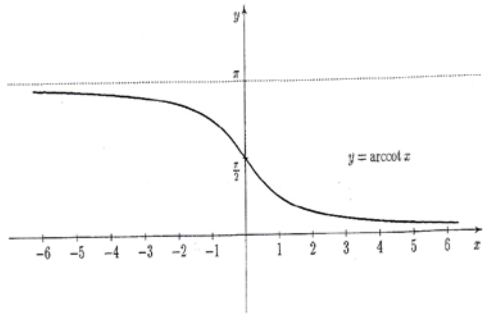

Fig. 16b

2.2. Limits of Functions

2.2.1. Definition

Intuitive idea of limit: The notation $\lim\limits_{x \to a} f(x) = A$

means that as x gets close to a (but does not equal a), $f(x)$ gets close to A (Fig. 17).

Definition of limit:

A function $y = f(x)$ is said to approach *a* limit *A* as *x* approaches the value *a* if, given small quantity ε, i tis possible to find a positive number δ such that $|f(x) - A| < \varepsilon$ for all *x* $|x - a| < \delta$, $x \neq a$, (Fig. 17).

Notation: $\boxed{\lim_{x\to a} f(x) = A}$

Symbolic notation of limit:

$\lim_{x\to a} f(x) = A \Leftrightarrow$ for $\forall \varepsilon > 0\ \exists\ \delta > 0$, that for $\forall\, x \in D(f)$ is valid: $|x - a| < \delta \Rightarrow |f(x) - A| < \varepsilon$.

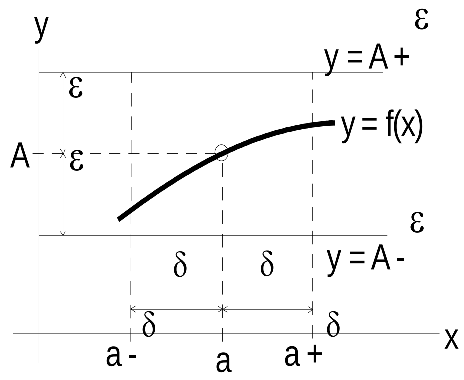

Fig. 17

Two facts regarding limits must be kept in mind:

a) The limit of a function as *x* approaches *a* is independent of the value of the function at *a*. Even though $\lim_{x\to a} f(x)$ exists, the value of the function at *a* may be undefined or may be the same as the limit or may be defined but different from the limit (Fig. 18a, b, c)

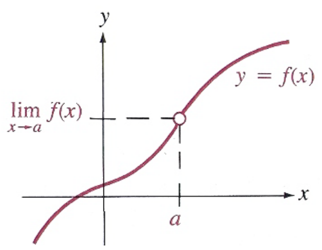

Fig. 18a

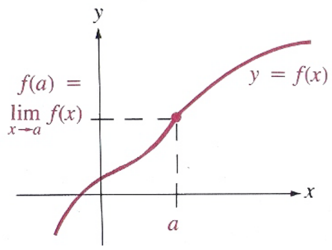

Fig. 18b

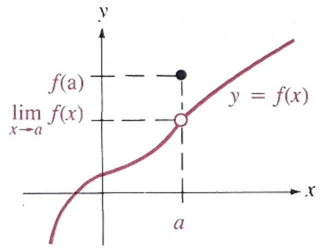

Fig. 18c

b) The limit is said to exist only if the following condition is satisfied:

The limit as *x* approaches *a* from left, written $\lim_{x\to a^-} f(x)$, equals the limit as *x* approaches *a* from right, written $\lim_{x\to a^+} f(x)$: $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

Fig 19a, b show two cases where $\lim_{x\to a} f(x)$ does not exist.

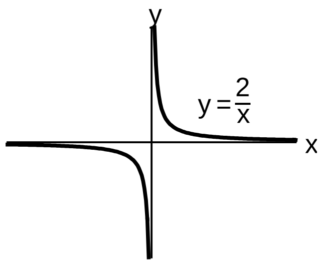

Fig. 19a

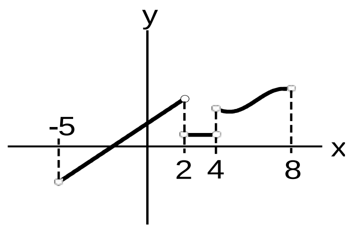

Fig.19b

2.2.2. Basic Rules

Suppose that $\lim\limits_{x\to a} f(x) = A$, $\lim\limits_{x\to a} g(x) = B$, than hold:

- There exists at most one $A\in R$ such that $\lim\limits_{x\to a} f(x) = A$,
- $\lim\limits_{x\to a} kf(x) = k \lim\limits_{x\to a} f(x) = kA$, $\quad k\in R$,
- $\lim\limits_{x\to a} [f(x)]^n = \left[\lim\limits_{x\to a} f(x)\right]^n = A^n$, $\quad n\in N$,
- $\lim\limits_{x\to a}\left[f(x)\pm g(x)\right] = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = A\pm B$,
- $\lim\limits_{x\to a}\left[f(x).g(x)\right] = \lim\limits_{x\to a} f(x) . \lim\limits_{x\to a} g(x) = A.B$,
- $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{A}{B}$ $\quad B\neq 0$.

2.2.3. Limits of Selected Functions

- $\lim\limits_{x\to a} c = c$, $\quad c\in R$,
- $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$,
- $\lim\limits_{x\to a} x^n = a^n$, $\quad a\in R,\ n\in N$,
- $\lim\limits_{x\to\pm\infty} \dfrac{k}{x^n} = 0$, $\quad n > 0$,
- $\lim\limits_{x\to+\infty} x^n = +\infty$, $\quad n\in N$,
- $\lim\limits_{x\to 0^-} \dfrac{1}{x} = -\infty$,
- $\lim\limits_{x\to 0^+} \dfrac{1}{x} = +\infty$, (Fig. 19a),
- $\lim\limits_{x\to-\infty} e^x = 0$,
- $\lim\limits_{x\to+\infty} e^x = +\infty$, (Fig. 11c),
- $\lim\limits_{x\to 0^+} \ln x = -\infty$,
- $\lim\limits_{x\to+\infty} \ln x = +\infty$, (Fig. 12c),
- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$,
- $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x = e$,

- $\lim\limits_{x\to\infty}\left(1+\dfrac{a}{x}\right)^x = e^a$.

Example 2.2.1. Find the limits: a) $A=\lim\limits_{x\to 3}\dfrac{x}{1-x}$, $B=\lim\limits_{x\to 2}\dfrac{2x-4}{x^2-4}$, c) $C=\lim\limits_{x\to\infty}\dfrac{x}{1-x^3}$,

d) $D=\lim\limits_{x\to+\infty}\dfrac{6x^2+1}{3x^2-2x+2}$, e) $E=\lim\limits_{x\to+\infty}\left(\dfrac{x+6}{x+2}\right)^x$.

Solution: a) $A=\dfrac{3}{1-3}=-\dfrac{3}{2}$,

b) $B=\lim\limits_{x\to 2}\dfrac{2(x-2)}{(x-2)(x+2)}=\lim\limits_{x\to 2}\dfrac{2}{x+2}=\dfrac{2}{2+2}=\dfrac{1}{2}$,

c) $C=\lim\limits_{x\to\infty}\dfrac{x}{1-x^3}=\lim\limits_{x\to\infty}\dfrac{x}{1-x^3}\dfrac{\frac{1}{x^3}}{\frac{1}{x^3}}=\lim\limits_{x\to\infty}\dfrac{\left(\frac{1}{x}\right)^2}{\left(\frac{1}{x}\right)^3-1}=\dfrac{0}{0-1}=0$,

d) $D=\lim\limits_{x\to+\infty}\dfrac{6x^2+1}{3x^2-2x+2}=\lim\limits_{x\to+\infty}\dfrac{6x^2+1}{3x^2-2x+2}\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim\limits_{x\to+\infty}\dfrac{6+\left(\frac{1}{x}\right)^2}{3-2\frac{1}{x}+2\left(\frac{1}{x}\right)^2}=\dfrac{6}{3}=2$,

e) $E=\lim\limits_{x\to+\infty}\left(\dfrac{x+6}{x+2}\right)^x=\lim\limits_{x\to+\infty}\left(\dfrac{x+2+4}{x+2}\right)^x==\lim\limits_{x\to+\infty}\left(1+\dfrac{4}{x+2}\right)^{x+2-2}=$

$=\lim\limits_{x\to+\infty}\left(1+\dfrac{4}{x+2}\right)=\lim\limits_{x\to+\infty}\left(1+\dfrac{4}{x+2}\right)^{x+2}.\left(1+\dfrac{4}{x+2}\right)^{-2}=e^4.1=e^4$.

2.3. Continuity of Functions

A function $y=f(x)$ is said to be **continuous at a point *a*** if $\boxed{\lim\limits_{x\to a} f(x)=f(a)}$**.**

Note that the definition requires three conditions to be satisfied:

a) $f(x)$ must be defined at the point a,

b) $f(x)$ must have a limit at point *a*,

c) this limit must be equal to the value $f(a)$.

We say that a function $f(x)$ is **continuous on open interval (*a*, *b*)**, if it is continuous at each point of the interval **(*a*, b).**

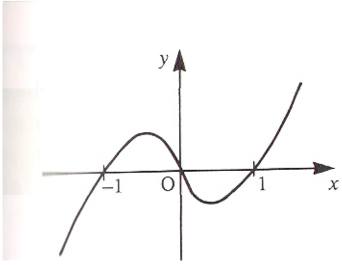

Fig. 20a

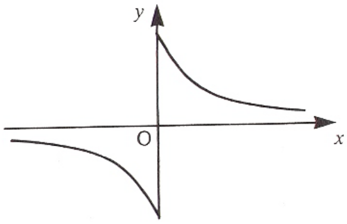

Fig. 20b

Consider the two functions (Fig. 20a, b). Function $f(x)$ in Fig. 20a we can draw the whole curve without lifting the pencil from the paper, but this is not possible for the function $g(x)$ in Fig. 20b. Hence the function $f(x)$ is continuous everywhere, while the function $g(x)$ is continuous on intervals $(-\infty, 0)$ and $(0, +\infty)$, and it has a discontinuity at $x = 0$.

Properties of continuous functions:

- Constant functions are continuous.
- Sums, differences, and products of continuous functions are continuous.
- Quotients and rational powers of continuous functions (where defined) are continuous.
- The composite function $f(g(x))$ is continuous at a if g is continuous at a and f is continuous at $g(a)$.

2.4. The Derivative of Function

2.4.1. Definition

Let $y = f(x)$ be a function and point $x_0 \in$D(f). The **derivative of the function f(x) at the point** x_0 is defined by limits

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{\Delta x \to 0} \frac{\Delta f(x_0)}{\Delta x}.$$

The derivative is denoted by symbols: $f'(x_0),\ y'(x_0),\ \dfrac{df(x_0)}{dx}, \dfrac{dy(x_0)}{dx}$.

If derivative $f'(x_0)$ is a finite number , we say that function $f(x)$ **is differentiable at** x_0**.**

Let $f(x)$ be a function differentiable at any point x from (a, b). Then we say that the function **is differentiable on interval (*a*, *b*).**

The function $f'(x)$ is called **the derivative function of** $f(x)$.

The process is called **differentiation**.

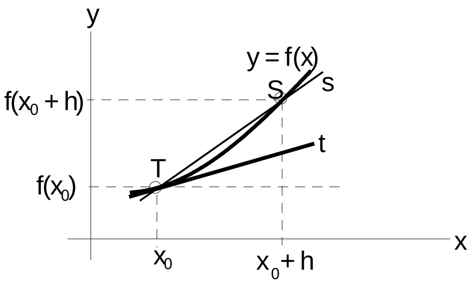

Fig. 21

Example 2.4.1: Find the derivative of the function $y = f(x) = c$ at point x_0.

Solution: $f'(x_0) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{c - c}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{0}{\Delta x} = \lim\limits_{\Delta x \to 0} 0 = 0.$

Example 2.4.2: Find the derivative of the function $y = x$ at point x_0.

Solution: $f'(x_0) = \lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{(x_0 + \Delta x) - (x_0)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x}{\Delta x} =$

$= \lim\limits_{\Delta x \to 0} 1 = 1.$

Geometric sense of derivative

The derivative $f'(x_0)$ is the slope k_t of the tangent *t* to the graph of function $y = f(x)$ at point $\mathrm{T}[x_0, y_0] = \mathrm{T}[x_0, f(x_0)]$: $k = k_t = f'(x_0) = \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$, (Fig. 21).

The tangent *t* to the graph of function $y = f(x)$ at point $= \mathrm{T}[x_0, f(x_0)]$ is described by equation

$$y - y_0 = k_t(x - x_0) \quad \text{or} \quad \boxed{y - f(x_0) = f'(x_0)(x - x_0)},$$

The normal line *n* to the graph of function $y = f(x)$ at point $= \mathrm{T}[x_0, f(x_0)]$ has slope $\boxed{k_n = -\dfrac{1}{k_t} = -\dfrac{1}{f'(x_0)}}$ (while $f'(x_0) \neq 0$) and it is described by equation

$$y - y_0 = k_n (x - x_0) \quad \text{or} \quad \boxed{y - f(x_0) = -\frac{1}{f'(x_0)}(x - x_0)}.$$

Example 2.4.3: Find the equations of the tangent and normal line of the function $y = x^2$ at point T[3, ?].

Solution: $y_0 = 3^2 = 9$, that is T[3, 9],

slope k_t of the tangent *t*: $y' = 2x, k_t = y'(3) = 2.3 = 6$,

the equation of the tangent *t*: $y - 9 = 6(x - 3)$,

slope k_n and the equation of the normal line *n*: $k_n = -\dfrac{1}{6}$, $\quad y - 9 = -\dfrac{1}{6}(x - 3)$.

Physical sense of derivative

The derivative $f'(t_0)$ is the instantaneous velocity v of physical point $T[t_0, f(t_0)]$ at time t_0:
$$v(t_0) = f'(t_0) = \lim_{h\to 0} \frac{f(t_0+h)-f(t_0)}{h}.$$

2.4.2. Basic Rules for differentiation

Let $f(x)$ and $g(x)$ be functions differentiable at point *x*, let $c\in R$ be a constant. Then the function $c.f(x)$, $f(x)\pm g(x)$, $f(x).g(x)$ and $\dfrac{f(x)}{g(x)}$, $(g(x)\neq 0)$ are differentiable at point *x*. The following hold:

- $((f(x)\pm g(x))' = f'(x)\pm g'(x)$,
- $(f(x).g(x))' = f'(x).g(x)+f(x).g'(x)$ and hence for $g(x)=c$: • $\left(c.f(x)\right)' = c.f'(x)$
- $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x).g(x)-f(x).g'(x)}{g^2(x)}$,
- the derivative of a composite function (the chain rule):

  Let $g(x)$ be a function differentiable at point *x* and *f* be a function differentiable at point $g(x)$, then the composite function $f(g(x))$ is differentiable at point *x*, and

  $(f(g(x)))' = f'(g(x)).g'(x)$.
- If a function $f(x)$ is differentiable at point x_0, then $f(x)$ is continuous at point x_0.
- *L´Hospital rule* (for limits of fraction with infinite denominator)

  Let for $a\in$R be $\lim\limits_{x\to a} f(x)=0$, $\lim\limits_{x\to a} g(x)=0$, respectively

  $\lim\limits_{x\to a} f(x)=\pm\infty$, $\lim\limits_{x\to a} g(x)=\pm\infty$, and $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists.

  Then also exists $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ and it holds $\boxed{\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}}$.

2.4.3. Derivatives of Selected Functions

- $(c)´ = c´= 0$, for $\forall x\in R$,
- $(x^n)' = nx^{n-1}$ for $\forall x\in R$, for $\forall n\in R$, hence • $(x)'=1$
- $(x^a)' = ax^{a-1}$ for $\forall x\in(0,+\infty)$, for $\forall a\in R$,
- $(e^x)' = e^x$ for $\forall x\in R$,
- $(a^x)' = a^x \ln a$ $\forall x\in R$, for $\forall a\in R$, $a>0$, $a\neq 1$,

- $\left(\ln|x|\right)' = \frac{1}{x}$ for $\forall x \in R$, $x \neq 0$,
- $(\log_a|x|)' = \frac{1}{x \ln a}$ for $\forall x \in R$, $x \neq 0$, for $\forall a \in R$, $a > 0$, $a \neq 1$,
- $(\sin x)' = \cos x$ for $\forall x \in R$,
- $(\cos x)' = -\sin x$ for $\forall x \in R$,
- $(\mathrm{tg}\, x)' = \frac{1}{\cos^2 x}$ for $\forall x \in R$, $x \neq (2k+1)\frac{\pi}{2}$,
- $(\mathrm{cotg}\, x)' = -\frac{1}{\sin^2 x}$ for $\forall x \in R$, $x \neq k\pi$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $\forall x \in (-1,1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $\forall x \in (-1,1)$,
- $(\arctan x)' = \frac{1}{1+x^2}$ for $\forall x \in R$,
- $(\mathrm{arc}\cot x)' = -\frac{1}{1+x^2}$ for $\forall x \in R$.

Example 2.4.4: Find the derivatives of the function at point $x \in D(f)$:

a) $y = 2x^4 + 3x^2 - 4x + 1$

Solution: $y' = 2.4x^{4-1} + 3.2x^{2-1} - 4.1 + 0 = 8x^3 + 6x - 4$.

b) $y = \frac{4}{x^3} + 5x^2 - \sqrt[4]{x^3}$

Solution: $y = 4x^{-3} + 5x^2 - x^{\frac{3}{4}}$,

$$y' = 4.(-3)x^{-3-1} + 5.2x^{2-1} - \frac{3}{4}x^{\frac{3}{4}-1} = -12x^{-4} + 10x - \frac{3}{4}x^{-\frac{1}{4}}\ \frac{3}{4}x^{\frac{3}{4}-1}$$

c) $y = x.\ln x$

Solution: $y' = (x)'.\ln x + x.(\ln x)' = 1.\ln x + x.\frac{1}{x} = \ln x + 1$.

d) $y = e^x \cos x$

Solution: $y' = (e^x)'.\cos x + e^x(\cos x)' = e^x \cos x + e^x - \sin x) = e^x(\cos x - \sin x)$

e) $y = (x^7 - 3x^2)\sin x$

Solution: $y' = \left(x^7 - 3x^2\right)'.\sin x + \left(x^7 - 3x^2\right).(\sin x)' = \left(7x^6 - 6x\right).\sin x + \left(x^7 - 3x^2\right).\cos x$.

e) $y = \dfrac{\ln x}{2x}$

Solution: y´ = $y' = \dfrac{(\ln x)'.2x - \ln x.(2x)'}{(2x)^2} = \dfrac{\frac{1}{x}2x - \ln x.2}{4x^2} = \dfrac{2 - 2\ln x}{4x^2} = \dfrac{2(1 - \ln x)}{4x^2} = \dfrac{1 - \ln x}{2x^2}$.

f) y = tg x = $\dfrac{\sin x}{\cos x}$

Solution: $y' = \dfrac{(\sin x)'.\cos x - \sin x.(\cos x)'}{(\cos x)^2} = \dfrac{\cos x.\cos x - \sin x.(-\sin x)}{(\cos x)^2} = \dfrac{1}{(\cos x)^2}$.

f) $y = e^{3x+5}$

Solution: $y = e^u$, $u = 3x + 5$: $y' = (e^u)'.u' = e^u.3 = e^{3x+5}.3 = 3e^{3x+5}$.

g) $y = \sin 3x$

Solution: $y = \sin u$, $u = 3x$: $y' = (\sin u)'.u' = \cos u.3 = \cos 3x.3 = 3\cos 3x$.

h) $y = \sin x^3$

Solution: $y = \sin u$, $u = x^3$: $y' = (\sin u)'.u' = \cos u.3x^2 = \cos x^3.3x^2 = 3x^2.\cos x^3$.

i) $y = \sin^3 x = (\sin x)^3$

Solution: $y = u^3$, $u = \sin x$: $y' = (u^3)'.u' = 3u^2.\cos x = 3\sin^2 x.\cos x$.

j) $y = (2x^5 - 2x - 1)^4$

Solution: $y = u^4$, $u = 2x^5 - 2x - 1$: $y' = (u^4)'.u' = 4u^3.(10x^4 - 2) =$

$= 4(2x^5 - 2x - 1)^3(10x^4 - 2) = 8(2x^5 - 2x - 1)^3(5x^4 - 1).$

i) $y = 3\arcsin(4x^2 + 1)$

Solution: $y = 3\arcsin u$, $u = 4x^2 + 1$: $y' = (3\arcsin u)'.u' = 3\dfrac{1}{\sqrt{1 - u^2}}.(8x) =$

$= 3\dfrac{1}{\sqrt{1 - (4x^2 + 1)^2}}.8x = \dfrac{24x}{\sqrt{1 - (4x^2 + 1)^2}}$.

2.4.4. Differential of the Function

Let $f(x)$ be a function differentiable at point x_0. **The differential of the function** $f(x)$ **at point** x_0 is called the linear function

$dy_0 = dy(x_0) = df(x_0) = f'(x_0).dx = y'(x_0).dx$, where $dx = x - x_0$.

The differential $dy(x)$ is used to describe a small change in the dependent variable *y*, and *dx* is a small change in the independent variable *x*.

2.4.5. Highes-Order Derivative

The derivative $y' = f'(x)$ of the function $f(x)$ is a function itself. We can compute the derivative $(y')' = (f'(x))'$ of the function $f'(x)$ f´(x), which is called *second derivative of the function* $f(x)$ and it is denoted by

$$y'',\ f''(x), \frac{d^2y}{dx^2},\ \frac{d^2f}{dx^2}.$$

Analogously $(y'')' = (f''(x))' = y''' = f'''(x) = \dfrac{d^3y}{dx^3} = \dfrac{d^3f}{dx^3}$
is called *third derivative of the function* $f(x)$, etc.

Generally the **n-th derivative of the function** $f(x)$ is defined by

$\left(y^{(n-1)}\right)' = \left(f^{(n-1)}(x)\right)'$ and denoted by $y^{(n)},\ f^{(n)}(x),\ \dfrac{d^n f(x)}{dx^n}, \dfrac{d^n y}{dx^n}$.

Example 2.4.5: Find all derivatives of the function $y = 5x^4 + 3x - 8$.

Solution: $y' = 20x^3 + 3$,, $y'' = 60x^2$, $y''' = 120x$,
$y^{(4)} = 120$, $y^{(5)} = y^{(6)} = \ldots = 0$.

Example 2.4.6: Find first and second derivative of the function $y = x\sin x$ at point $x_0 = 0$.

Solution: $y' = 1.\sin x + x.\cos x$, $y'(0) = 1.\sin 0 + 0.\cos 0 = 1.0 + 0.1 = 0$,
$y''(0) = 2\cos 0 - 0.\sin 0 = 2.1 - 0.0 = 2$.

2.4.6. Parametric Differentiation

We say that a function is defined **parametrically** by $y = f(x)$, where
$x = x(t)$ and $y = y(t)$, $t \in \langle a, b\rangle$ is parameter.

For a derivative of this function is hold: $\boxed{y' = f'(x) = \dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\dot{y}}{\dot{x}}}$.

Example 2.4.7: Find a derivative of the function $x = a(t - \sin t),\ y = a(1 - \cos t),\ a > 0,\ t \in R$.

Solution: $\dot{x} = a(1 - \cos t),\ \dot{y} = a\sin t$ and then $y' = \dfrac{a\sin t}{a(1-\cos t)} = \dfrac{\sin t}{(1-\cos t)}$.

2.5. Applications of the Derivatives

2.6.1. Basic Theorems

The Mean Value Theorem

Let $y = f(x)$ be a function differentiable on (a,b) and continuous on $<a,b>$. Then there is at least one point $c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$, Fig. 22.

Fig. 22 Fig. 23

Rolle´s Theorem

Let $y = f(x)$ be a function differentiable on (a,b) and continuous on $<a,b>$. If $f(a) = f(b)$, then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$, Fig. 23.

Cauchy´s Theorem

Let functions $f(x)$ and $g(x)$ be continuous on $<a,b>$, function $f(x)$ be differentiable on interval (a, b), function $g(x)$ has a finite derivative $g'(x) \neq 0$ on (a,b). Then there exists a point $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$.

2.5.6. Monotonic Functions and Optimal Value

From the Mean Value Theorem there is valid: Suppose that two functions $f(x)$ and $g(x)$ are differentiable on (a,b) and continuous and bounded on $<a,b>$. Then the following statements are true:

If $f'(x) > 0$ for $\forall x \in (a,b)$, then $f(x)$ is increasing on (a,b),

if $f'(x) < 0$ for $\forall x \in (a,b)$, then $f(x)$ is decreasing on (a,b),

if $f'(x) \geq 0$ for $\forall x \in (a,b)$, then $f(x)$ is nondecreasing on (a,b),

if $f'(x) \leq 0$ for $\forall x \in (a,b)$, then $f(x)$ is nonincreasing on (a,b),

if $f'(x) = 0$ for $\forall x \in (a,b)$, then $f(x)$ is constant on (a,b).

Definition:

1. A function $f(x)$ with domain $D(f)$ is said to have an **absolute maximum** (respectively **absolute minimum**) **at point** $x_0 \in D(f)$ if $f(x) \leq f(x_0)$, (respectively if $f(x) \geq f(x_0)$) for $\forall x \in D(f)$**.** The number $f(x_0)$) is called the absolute maximum (respectively absolute minimum) of $f(x)$ on $D(f)$.
2. The function $f(x)$ is said to have a **local** or **relative maximum** (respectively **local** or **relative minimum**) **at point** $x_0 \in D(f)$ if there is some open interval $(a,b) \subseteq D(f)$ containing x_0 and $f(x_0)$ is the absolute maximum (respectively absolute minimum) of

$f(x)$ on (a,b)**.** The number $f(x_0)$ is called a local or relative maximum (respectively local or relative minimum) of $f(x)$ on (a,b)**.**

3 An absolute maximum or absolute minimum of $f(x)$ is called an **absolute extreme of f(x).** A local maximum or local minimum of $f(x)$ is called a **local extreme of** $f(x)$, Fig 24a,b).

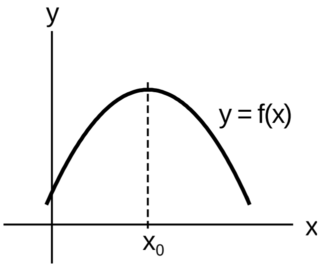

Fig. 24a

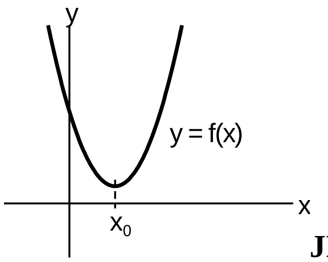

Fig. 24b

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Proposition:

1. If a function $f(x)$ has a local extreme at point x_0, then
   either $f'(x) = 0$, or $f'(x)$ does not exist.

   (When searching for local extremes of a function $f(x)$, in view of this result , it suffices to test only those points x_0, for which $f'(x_0) = 0$ or $f'(x_0)$ does not exist. These points are called **critical** or **stationary point**.)

2. **The first derivative test for extreme**

   Let $y = f(x)$ be a function continuous on (a,b) and critical point $x_0 \in (a,b)$.

   If $f'(x) > 0$ on (a, x_0) and $f'(x) < 0$ on (x_0, b), then at point x_0 there is a local maximum $f(x_0)$ of the function f(x) on (a,b).

   If $f'(x) < 0$ on (a, x_0) and $f'(x) > 0$ on (x_0, b), then at point x_0 there is a local minimum $f(x_0)$ of the function $f(x)$ on (a,b).

   The second derivative test for extreme

   Suppose that $f(x)$, $f'(x)$, $f''(x)$ exist on (a,b) and $x_0 \in (a,b)$. Let $f'(x_0) = 0$**.**

   Then the following statements are true:

   If $f''(x) > 0$, then at point x_0 there is a local minimum $f(x_0)$ of the function $f(x)$ on (a,b).

   If $f''(x) < 0$, then at point x_0 there is a local maximum $f(x_0)$ of the function $f(x)$ on (a,b).

Determination of absolute maximum or absolute minimum of function $f(x)$ on $< a,b >$**:**

Every function $f(x)$ continuous on $< a,b >$ attains both its absolute maximum and absolute minimum there. Therefore, if we determine absolute maximum and absolute minimum of function $f(x)$, we proceed as follows:

a) Find critical points of $f(x)$.

b) Compute the values of $f(x)$ at all critical points and $f(a), f(b)$.

c) The largest value among them is absolute maximum, the least value among them is absolute minimum of the function $f(x)$ on $<a,b>$.

2.5.3. Convexity and Concavity of a Function

We say, that a function $f(x)$ is **convex at point** x_0, if there exists an interval $(x_0-\delta, x_0+\delta)$ such that the graph of the function $f(x)$ restricted to $(x_0-\delta, x_0+\delta)$ lies above the tangent drawn at the point [$[x_0, f(x_0)]$, Fig. 25a. If $f(x)$ is convex at every point of (a,b), we say that $f(x)$ is **convex** (or concave up or concave upward) on (a,b).

We say, that a function $f(x)$ is **concave at point** x_0, if there exists a interval $(x_0-\delta, x_0+\delta)$ such that the graph of the function $f(x)$ restricted to $(x_0-\delta, x_0+\delta)$ lies below the tangent drawn at the point $[x_0, f(x_0)]$, Fig. 25b. If $f(x)$ is concave at every point of (a,b), we say that $f(x)$ is **concave** (or concave down or concave downward) on (a,b).

We say, that a point $[x_0, f(x_0)]$ is a **point of inflection** (inflection point) of a function $f(x)$ if there exists some $\delta > 0$ such that either the graph of $f(x)$ is convex on $(x_0-\delta, x_0)$ and concave down on $(x_0, x_0+\delta)$, Fig. 25c, or the graph of $f(x)$ is concave down on $(x_0-\delta, x_0)$ and convex on $(x_0, x_0+\delta)$.

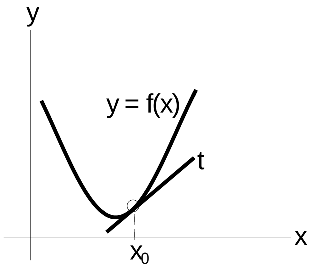

Fig. 25a

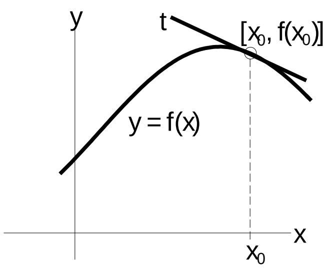

Fig. 25b

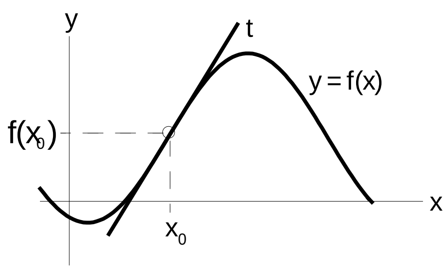

Fig. 25c

Proposition: Test of convexity and concavity and inflection point

Suppose that $f''(x)$ of the function $f(x)$ exists on (a, b):

If $f''(x) > 0$ for $\forall x \in (a,b)$, then $f(x)$ is convex on (a,b),

if $f''(x) < 0$ for $\forall x \in (a,b)$, then $f(x)$ is concave on (a,b),

if $f''(x_0) = 0$ and $f'''(x_0) \neq 0$ then $[x_0, f(x_0)]$, is an inflection point of $f(x)$.

Example 2.5.1: Draw a graph of the function $y = \dfrac{x^2+1}{x}$.

Solution: a) $x \neq 0$, than domain $D(f) = (-\infty, 0) \cup (0, +\infty) = R - \{0\}$.

b) $y(-x) = \dfrac{(-x)^2+1}{-x} = -\dfrac{x^2+1}{x} = -y$, therefore the function is odd (its graph is symmetrical about the origin).

c) $x^2+1 \neq 0$ for $\forall x \in R$, therefore there is no point of intersection with x-axis.
$0 \notin D(f)$, therefore there is no point of intersection with y-axis too.

d) $y' = \dfrac{2x.x-(x^2+1).1}{x^2} = \dfrac{x^2-1}{x^2}$,

critical point: $y' = \dfrac{x^2-1}{x^2} = 0$, $x^2-1=0$, $x_1 = 1, x_2 = -1$.

The function is increasing for $y' > 0$: $\dfrac{x^2-1}{x^2} > 0$, $x^2-1 > 0$, $|x| > 1$,
$x \in (-\infty,-1) \cup (1,+\infty)$,

The function is decreasing for $y' < 0$: $\dfrac{x^2-1}{x^2} < 0$, $x^2-1 < 0$, $|x| < 1$, $x \in (-1,+1)$ - $\{0\}$.

At point $x_1 = 1$ there is a local minimum: $y(1) = 2$,
At point $x_2 = -1$ there is a local maximum , $y(-1) = -2$.

e) $y'' = \dfrac{2x.x^2-(x^2-1).2x}{x^4} = \dfrac{2x}{x^4} = \dfrac{2}{x^3}$,

equation $y'' = \dfrac{2}{x^3} = 0$ has no solution, therefore a function has no inflection point.

The function is convex for $y'' > 0$: $\dfrac{2}{x^3} > 0$, $x^3 > 0$, $x > 0$, $x \in (0,+\infty)$,

The function is concave for $y'' < 0$: $\dfrac{2}{x^3} < 0$, $x^3 < 0$, $x < 0$, $x \in (-\infty, 0)$.

f) The graph of the function $y = \dfrac{x^2+1}{x}$ is drawn on Fig. 26.

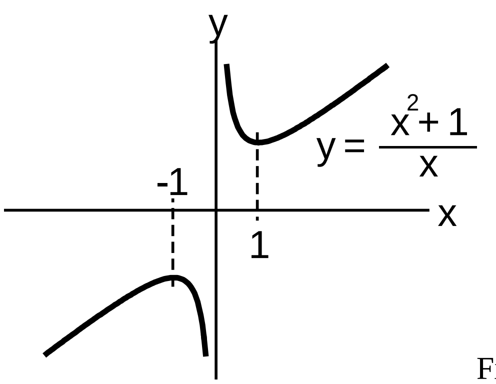

Fig. 26.

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